Online Supplement to "Common Bubble Detection in Large Dimensional Financial Systems"*

Ye Chen†, Peter C.B. Phillips††, Shuping Shi†††

†Capital University of Economics and Business

††Yale University, University of Auckland, University of Southampton, Singapore Management University

†††Macquarie University

October 9, 2019

This document is an Online Supplement to Chen, Phillips & Shi (2019). Section A provides the proofs of Lemma 4.1 in Chen, Phillips & Shi (2019) which states the consistency of the estimated factor under the null. The arguments follow directly from Bai (2004) and Chen, Li & Phillips (2019). Section B presents the Monte Carlo simulation results for the real time PSY-factor procedure.

A Proof of Lemma 4.1

We introduce the main notation again for ease of reference. The data generating process under the null is

$$X = F^0\Lambda^{0\prime} + E, \tag{1}$$

where $X = (\underline{X}_1 \ldots, \underline{X}_N)$ is a $T \times N$ matrix of the observed data with $\underline{X}_i = (X_{i1}, \ldots, X_{iT})'$, $F^0 = (f_{0,1} \ldots, f_{0,T})'$ is a $T \times 1$ vector, $\Lambda^0 = (\lambda_{0,1} \ldots, \lambda_{0,N})'$ is a $N \times 1$ vector of loading coefficients, and $E = (\underline{e}_1 \ldots, \underline{e}_N)$ is a $T \times N$ matrix of idiosyncratic errors with $\underline{e_i} = (e_{i1}, \ldots, e_{iT})'$. Alternatively,

$$X_t = \Lambda^0 f_{0,t} + e_t, \tag{2}$$

where $X_t = (X_{1t}, \ldots, X_{Nt})'$ and $e_t = (e_{1t}, \ldots, e_{Nt})'$.

The common factor F^0 and the factor loadings Λ^0 are estimated by the principal component analysis, subject to a normalization on the loadings. The estimated loadings for the first factor is denoted by $\tilde{L}_1 = \left(\tilde{l}_{1,1}, \tilde{l}_{1,2}, \ldots, \tilde{l}_{1,N}\right)'$, which is $\sqrt{N}$ times the eigenvectors corresponding to the largest eigenvalue (denoted by λ) of the $N \times N$ matrix $X'X$. The estimated first common factor is denoted by $\tilde{y}$ and defined as

$$\tilde{y} = X\tilde{L}_1/N.$$

Let λ_{NT} be the largest eigenvalues of $\frac{1}{NT^2}X'X$. By the definition of eigenvectors and eigenvalues, we have the mathematical equality

$$\frac{1}{NT^2}X'X\tilde{L}_1 = \tilde{L}_1\lambda_{NT}. \tag{3}$$

*Chen acknowledges support from National Natural Science Foundation of China (No.71803138). Phillips acknowledges support from the Kelly Fund, University of Auckland, the NSF under Grant No. SES 18-50860, and a LKC Fellowship at Singapore Management University. Shi acknowledges support from the Australian Research Council Discovery Projects funding scheme (project number DP190102049). Ye Chen, International School of Economics and Management, Capital University of Economics and Business; Email: zoeyechen_cueb@163.com. Peter C.B. Phillips, Yale University, University of Auckland, University of Southampton & Singapore Management University; Email: peter.phillips@yale.edu. Shuping Shi, Department of Economics, Macquarie University; Email: shuping.shi@mq.edu.au.

It follows that

$$\tilde{L}_1 = \frac{1}{NT^2} X' X \tilde{L}_1 \lambda_{NT}^{-1} = \frac{1}{NT^2} \left(\Lambda^0 F^{0\prime} F^0 \Lambda^{0\prime} + \Lambda^0 F^{0\prime} E + E' F^0 \Lambda^{0\prime} + E' E \right) \tilde{L}_1 \lambda_{NT}^{-1} \tag{4}$$

using (1). Let $H = \lambda_{NT}^{-1} \frac{F^{0\prime} F^0}{T^2} \frac{\Lambda^{0\prime} \tilde{L}_1}{N}$ such that $H \Lambda^0 = \frac{1}{NT^2} \Lambda^0 F^{0\prime} F^0 \Lambda^{0\prime}$. We have

$$\begin{aligned} & \tilde{l}_{1,i} - H \lambda_{i,0} \\ = \ & \lambda_{NT}^{-1} \left[\frac{T}{NT^2} \sum_{j=1}^{N} \tilde{l}_{1,j} E \left(\frac{\underline{e}_i' \underline{e}_j}{T} \right) + \frac{T}{NT^2} \sum_{j=1}^{N} \tilde{l}_{1,j} \left(\frac{\underline{e}_i' \underline{e}_j}{T} - \frac{E\left(\underline{e}_i' \underline{e}_j \right)}{T} \right) \right. \\ & \left. \frac{1}{NT^2} \lambda_{0,i} \sum_{j=1}^{N} \tilde{l}_{1,j} F^{0\prime} \underline{e}_j + \frac{1}{NT^2} \underline{e}_i' F^0 \sum_{j=1}^{N} \lambda_{0,j} \tilde{l}_{1,j} \right] \\ = \ & \lambda_{NT}^{-1} \left[\frac{1}{NT} \sum_{j=1}^{N} \tilde{l}_{1,j} \gamma_T(i,j) + \frac{1}{NT} \sum_{j=1}^{N} \tilde{l}_{1,j} \zeta_{ij} + \frac{1}{N} \sum_{j=1}^{N} \tilde{l}_{1,j} \eta_{ij} + \frac{1}{N} \sum_{j=1}^{N} \tilde{l}_{1,j} \xi_{ij} \right], \end{aligned} \tag{5}$$

where $\gamma_T(i,j) = E\left(\underline{e}_i' \underline{e}_j \right)/T$, $\zeta_{ij} = \underline{e}_i' \underline{e}_j / T - \gamma_T(i,j)$, $\eta_{ij} = \lambda_{0,i} F^{0\prime} \underline{e}_j / T^2$, and $\xi_{ij} = \underline{e}_i' F^0 \lambda_{0,j} / T^2$.

A.1 Auxilary Lemmas

Lemma S.1, S.2 and S.3 are given here to aid the proof of Lemma 4.1.

Lemma S.1. *Under the date generating process (1) and Assumptions 4.1-4.5, as $N, T \to \infty$, we have*

(1) $\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_T(i,j)^2 \le M$;

(2) $\lambda_{NT} = O_p(1)$;

(3) $H = \lambda_{NT}^{-1} \frac{F^{0\prime} F^0}{T^2} \frac{\Lambda^{0\prime} \tilde{L}_1}{N} = O_p(1)$;

Proof . *(1) The proof is as in Chen, Li & Phillips (2019) and is given here for convenience. Let* $\rho_{i,j} = \gamma_T(i,j) / [\gamma(i,i)\gamma(j,j)]^{1/2}$ *and by construction* $\rho_{i,j}^2 \le |\rho_{i,j}| \le 1$. *Since* $\gamma_T(i,i) \le M$,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_T(i,j)^2 &= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{i,j}^2 \gamma_T(i,i) \gamma_T(j,j) \\ &\le M \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\rho_{i,j}| \left| \gamma_T(i,i) \gamma_T(j,j) \right|^{1/2} \\ &\le M \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\gamma_T(i,j)| \le M^2 \end{aligned}$$

by Assumption 4.3(2).

(2) λ_{NT} *denotes the leading eigenvalue of* $\frac{1}{NT^2} X' X$ *and is* $O_p(1)$. *The result is from Bai (2004, Lemma B.3) and Lemma A.3 in Chen, Li & Phillips (2019). A different proof is given here. From the definition of the eigenvalues and eigenvectors and the normalization* $\tilde{L}_1' \tilde{L}_1 / N = 1$, *we have*

$$\begin{aligned} \lambda_{NT} &= \frac{1}{N^2 T^2} \tilde{L}_1' X' X \tilde{L}_1 \\ &= \frac{1}{N^2 T^2} \tilde{L}_1' \left(\Lambda^0 F^{0\prime} F^0 \Lambda^{0\prime} + \Lambda^0 F^{0\prime} E + E' F^0 \Lambda^{0\prime} + E' E \right) \tilde{L}_1. \end{aligned}$$

The first term

$$\frac{1}{N^2T^2}\tilde{L}_1'\Lambda^0 F^{0\prime}F^0\Lambda^{0\prime}\tilde{L}_1 = \frac{1}{N^2}\left(\frac{F^{0\prime}F^0}{T^2}\right)\left(\sum_{i=1}^{N} l_{1,i}\lambda_{0,i}\right)^2 = O_p(1),$$

is from Lemma A.2 and by Cauchy-Schwartz inequality, we obtain

$$\sum_{i=1}^{N}\tilde{l}_{1,i}\lambda_{0,i} \leq N\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{1,i}^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{0,i}^2\right)^{1/2} = O_p(N). \tag{6}$$

using Assumption 4.2(1). The second term

$$\begin{aligned}
\frac{1}{N^2T^2}\tilde{L}_1'\Lambda^0 F^{0\prime}E\tilde{L}_1 &= \frac{1}{N^2T^2}\left(\sum_{i=1}^{N}\tilde{l}_{1,i}\lambda_{0,i}\right)\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{l}_{1,i}f_{0,t}e_{it}\right)\\
&= \frac{1}{N^2T}\left(\sum_{i=1}^{N}\tilde{l}_{1,i}\lambda_{0,i}\right)\sum_{i=1}^{N}\tilde{l}_{1,i}\left(\frac{1}{T}\sum_{t=1}^{T}f_{0,t}e_{it}\right)\\
&= O_p\left(\frac{1}{T}\right)
\end{aligned}$$

from Assumption 4.4, equation (6), and

$$\left(\sum_{i=1}^{N}\tilde{l}_{1,i}\right)^2 \leq \left(\sum_{i=1}^{N}\tilde{l}_{1,i}^2\right)\left(\sum_{i=1}^{N}1\right) = N\sum_{i=1}^{N}\tilde{l}_{1,i}^2 = N^2\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{1,i}^2\right) = O_p\left(N^2\right). \tag{7}$$

by the Cauchy-Schwartz inequality. The third term is $O_p\left(\frac{1}{T}\right)$ since

$$\frac{1}{N^2T^2}\tilde{L}_1'E'F^0\Lambda^{0\prime}\tilde{L}_1 = \frac{1}{N^2T^2}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\tilde{l}_{1,i}f_{0,t}e_{it}\right)\left(\sum_{i=1}^{N}\tilde{l}_{1,i}\lambda_{0,i}\right) = O_p\left(\frac{1}{T}\right).$$

The fourth term is $O_p\left(\frac{1}{T}\right)$ since

$$\frac{1}{N^2T^2}\tilde{L}_1'E'E\tilde{L}_1 = \frac{1}{N^2T^2}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\tilde{l}_{1,i}e_{it}\right)^2 \leq \frac{1}{T^2}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{1,i}^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}e_{it}^2\right) = O_p\left(\frac{1}{T}\right).$$

Thus, the first term dominates the others and we establish

$$\lambda_{NT} = \frac{1}{N^2T^2}\tilde{L}_1'\Lambda^0 F^{0\prime}F^0\Lambda^{0\prime}\tilde{L}_1\left[1+o_p(1)\right] = O_p(1).$$

(3) The normalization variable

$$H = \lambda_{NT}^{-1}\frac{F^{0\prime}F^0}{T^2}\frac{\Lambda^{0\prime}\tilde{L}_1}{N} = \frac{1}{N}\lambda_{NT}^{-1}\frac{F^{0\prime}F^0}{T^2}\left(\sum_{i=1}^{N}\tilde{l}_{1,i}\lambda_{0,i}\right) = O_p(1),$$

from Lemma S1(2), Lemma A.2(1) and (6).

Lemma S.2. *Under the data generating process (1) and Assumptions 4.1-4.4, as $N,T\to\infty$, we have*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i}-H\lambda_{0,i}\right)^2 = O_p\left(T^{-2}\right).$$

Proof . *(1) Using (5) and the inequality* $(x+y+z+u)^2 \le 4\left(x^2+y^2+z^2+u^2\right)$, *we have*

$$\left(\tilde{l}_{1,i} - H\lambda_{i,0}\right)^2 \le 4\left(a_i^2 + b_i^2 + c_i^2 + d_i^2\right),$$

where

$$\begin{aligned} a_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^2T^2}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\gamma_T(i,j)\right)^2, \; b_i^2 = \lambda_{NT}^{-2}\frac{1}{N^2T^2}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\zeta_{ij}\right)^2, \\ c_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^2}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\eta_{ij}\right)^2, \text{ and } d_i^2 = \lambda_{NT}^{-2}\frac{1}{N^2}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\xi_{ij}\right)^2. \end{aligned}$$

It follows that

$$\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i} - H\lambda_{i,0}\right)^2 \le 4\left(\frac{1}{N}\sum_{i=1}^{N}a_i^2 + \frac{1}{N}\sum_{i=1}^{N}b_i^2 + \frac{1}{N}\sum_{i=1}^{N}c_i^2 + \frac{1}{N}\sum_{i=1}^{N}d_i^2\right).$$

The first term, by the Cauchy-Schwarz inequality, is

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N}a_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^3T^2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\gamma_T(i,j)\right)^2 \\ &\leqslant \frac{1}{NT^2}\lambda_{NT}^{-2}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}^2\right)\left(\frac{1}{N}\sum_{j=1}^{N}\gamma_T(i,j)^2\right) \\ &= \frac{1}{NT^2}\lambda_{NT}^{-2}\left(\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_T(i,j)^2\right) \\ &= O_p\left(N^{-1}T^{-2}\right), \end{aligned}$$

from Lemma S.1(1) and Lemma S.1(2) and the normalization $\frac{1}{N}\tilde{L}_1'\tilde{L}_1 = 1$. *The second term is*

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N}b_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^3T^2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\zeta_{ij}\right)^2 \\ &= \lambda_{NT}^{-2}\frac{1}{N^3T^2}\sum_{i=1}^{N}\sum_{j,u=1}^{N}\tilde{l}_{1,j}\tilde{l}_{1,u}\zeta_{ij}\zeta_{iu} \\ &= \lambda_{NT}^{-2}\frac{1}{N^3T^2}\sum_{j,u=1}^{N}\left(\tilde{l}_{1,j}\tilde{l}_{1,u}\right)\left(\sum_{i=1}^{N}\zeta_{ij}\zeta_{iu}\right) \\ &\le \lambda_{NT}^{-2}\frac{1}{N^3T^2}\left[\sum_{j,u=1}^{N}\left(\tilde{l}_{1,j}\tilde{l}_{1,u}\right)^2\right]^{1/2}\left[\sum_{j,u=1}^{N}\left(\sum_{i=1}^{N}\zeta_{ij}\zeta_{iu}\right)^2\right]^{1/2} \\ &\le \lambda_{NT}^{-2}\frac{1}{NT^2}\left(\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}^2\right)\left[\frac{1}{N^2}\sum_{j,u=1}^{N}\left(\sum_{i=1}^{N}\zeta_{ij}\zeta_{iu}\right)^2\right]^{1/2} \\ &= \lambda_{NT}^{-2}\frac{1}{NT^2}\left[\frac{1}{N^2}\sum_{j,u=1}^{N}\left(\sum_{i=1}^{N}\zeta_{ij}\zeta_{iu}\right)^2\right]^{1/2}. \end{aligned}$$

We know that

$$E\left(\sum_{i=1}^{N}\zeta_{ij}\zeta_{iu}\right)^2 = E\left(\sum_{i,s=1}^{N}\zeta_{ij}\zeta_{iu}\zeta_{sj}\zeta_{su}\right) \leqslant N^2\max E\left|\zeta_{ij}\right|^4,$$

and

$$E\left|\zeta_{ij}\right|^4=E\left|\frac{\underline{e}_i'\underline{e}_j}{T}-\gamma_T\left(i,j\right)\right|^4=\left(\frac{1}{T^2}\right)E\left|T^{1/2}\left(\frac{\underline{e}_i'\underline{e}_j}{T}-\gamma_T\left(i,j\right)\right)\right|^4\leqslant\frac{1}{T^2}M,$$

from Assumption 4.3 (5). Thus,

$$\frac{1}{N}\sum_{i=1}^{N}b_i^2=O_p\left(T^{-3}\right).$$

The third term is

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}c_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^3}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\eta_{ij}\right)^2\\ &\leq \lambda_{NT}^{-2}\frac{1}{N^2}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}^2\right)\left(\sum_{j=1}^{N}\eta_{ij}^2\right)\\ &= \lambda_{NT}^{-2}\frac{1}{N}\left[\frac{1}{T^2}\sum_{j=1}^{N}\left(\frac{1}{T}F^{0\prime}\underline{e}_j\right)^2\right]\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{0,i}^2\right)\\ &= O_p\left(T^{-2}\right),\end{aligned}$$

from Assumption 4.2(1) and Lemma S.1 (1) and by virtue of Assumption 4 since

$$\left[\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{T}F^{0\prime}\underline{e}_j\right)^2\right]\leqslant\left(sup_{j\geqslant1}\left|\frac{1}{T}F^{0\prime}\underline{e}_j\right|\right)^2=O_p(1).$$

Similarly, the last term is

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}d_i^2 &= \lambda_{NT}^{-2}\frac{1}{N^3}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\xi_{ij}\right)^2\\ &= \lambda_{NT}^{-2}\frac{1}{N^3}\frac{1}{T^4}\left(\sum_{j=1}^{N}\tilde{l}_{1,j}\lambda_{0,j}\right)^2\left(\sum_{i=1}^{N}\left(\underline{e}_i'F^0\right)^2\right)\\ &= O_p\left(T^{-2}\right),\end{aligned}$$

from (6) *and Assumption 4.4. Therefore, the third and fourth terms dominate and*

$$\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i}-H\lambda_{0,i}\right)^2=O_p\left(T^{-2}\right).$$

Lemma S.3. *Under the data generation (1) and Assumption 4.1-4.5, as $T,N\to\infty$,*

$$\Lambda^{0\prime}\left(\Lambda^0H-\tilde{L}_1\right)/N=O_p\left(\frac{1}{T\sqrt{N}}\right).$$

Proof . *(i) Using (5), we have*

$$\begin{aligned}&\Lambda^{0\prime}\left(\Lambda^0H-\tilde{L}_1\right)/N\\ =&-\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i}-H\lambda_{0,i}\right)\lambda_{0,i}\\ =&-\frac{1}{N}\lambda_{NT}^{-1}\sum_{i=1}^{N}\left[\frac{1}{NT^2}\sum_{j=1}^{N}\tilde{l}_{1,j}\underline{e}_i'\underline{e}_j+\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}\eta_{ij}+\frac{1}{N}\sum_{j=1}^{N}\tilde{l}_{1,j}\xi_{ij}\right]\lambda_{0,i}\\ =&-H\lambda_{NT}^{-1}\left[\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{0,j}\underline{e}_i'\underline{e}_j\lambda_{0,i}+\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{0,j}\eta_{ij}\lambda_{0,i}+\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{0,j}\xi_{ij}\lambda_{0,i}\right]\left[1+o_p\left(1\right)\right]\\ =&\ I+II+III\end{aligned}$$

where I,II,III correspond to the three component sums. The second to last equality is due to the fact that

$$\tilde{l}_{1,j} = (\tilde{l}_{1,j} - H\lambda_{0,j}) + H\lambda_{0,j} = H\lambda_{0,j}\left[1 + o_p(1)\right],$$

which follows directly from the proof of Lemma S.2. The first term I is

$$-H\lambda_{NT}^{-1}\frac{1}{N^2T^2}\sum_{i=1}^{N}\lambda_{0,i}\underline{e}_i'\sum_{j=1}^{N}\lambda_{0,j}\underline{e}_j = -\frac{1}{NT}H\lambda_{NT}^{-1}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\lambda_{0,j}e_{jt}\right)^2\right] = O_p\left(N^{-1}T^{-1}\right),$$

by Lemma 1(ii) in Bai and Ng (2002). The second term II is

$$-\lambda_{NT}^{-1}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}H\lambda_{0,j}\eta_{ij}\lambda_{0,i} = -\frac{1}{NT^2}H\lambda_{NT}^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{0,i}^2\right)\sum_{t=1}^{T}\sum_{j=1}^{N}\lambda_{0,j}f_{0,t}e_{jt} = O_p\left(\frac{1}{\sqrt{N}T}\right),$$

since

$$\sum_{t=1}^{T}\sum_{j=1}^{N}\lambda_{0,j}f_{0,t}e_{jt} = T\sum_{i=1}^{N}\lambda_{0,i}\left(\frac{1}{T}\sum_{t=1}^{T}f_{0,t}e_{it}\right) = O_p\left(T\sqrt{N}\right).$$

The third term III is

$$-\lambda_{NT}^{-1}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}H\lambda_{0,j}\xi_{ij}\lambda_{0,i} = -H\lambda_{NT}^{-1}\frac{1}{NT^2}\left(\frac{1}{N}\sum_{j=1}^{N}\lambda_{0,j}^2\right)\sum_{i=1}^{N}\underline{e}_i'F^0\lambda_{0,i} = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

Thus, we have

$$\Lambda^{0\prime}\left(\Lambda^0 H - \tilde{L}_1\right)/N = I + II + III = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

Lemma 4.1. *Under Assumptions 4.1-4.5, as $N,T\to\infty$, we have*

$$\delta_{NT}^2\left(\frac{1}{T}\sum_{t=1}^{T}\left|\tilde{y}_t - H^0 f_{0,t}\right|^2\right) = O_p(1), \tag{8}$$

where $\delta_{NT} = \min\left(\sqrt{N},T\right)$ and $H^0 = H^{-1}$.

Proof . *By construction,*

$$\tilde{y}_t = \tilde{L}_1'X_t/N = \tilde{L}_1'\left(\Lambda^0 f_{0,t} + e_t\right)/N.$$

Rewriting $\Lambda^0 = \Lambda^0 - \tilde{L}_1H^{-1} + \tilde{L}_1H^{-1}$ and using the restriction $\tilde{L}_1'\tilde{L}_1/N = 1$, we have

$$\tilde{y}_t = H^{-1}f_{0,t} + \tilde{L}_1'\left(\Lambda^0 - \tilde{L}_1H^{-1}\right)f_{0,t}/N + \left(\tilde{L}_1' - H\Lambda^{0\prime}\right)e_t/N + H\Lambda^{0\prime}e_t/N$$

As before, we use the inequality

$$\left(\tilde{y}_t - H^{-1}f_{0,t}\right)^2 \le 4\left(a_t^2 + b_t^2 + c_t^2\right),$$

where

$$a_t^2 = \frac{1}{N^2}\left[\tilde{L}_1'\left(\Lambda^0 - \tilde{L}_1H^{-1}\right)f_{0,t}\right]^2, b_t^2 = \frac{1}{N^2}\left[\left(\tilde{L}_1' - H\Lambda^{0\prime}\right)e_t\right]^2, \text{ and } c_t^2 = \frac{1}{N^2}\left(H\Lambda^{0\prime}e_t\right)^2.$$

It follows that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\tilde{y}_t - H^{-1}f_{0,t}\right)^2 \le 4\left(\frac{1}{T}\sum_{t=1}^{T}a_t^2 + \frac{1}{T}\sum_{t=1}^{T}b_t^2 + \frac{1}{T}\sum_{t=1}^{T}c_t^2\right).$$

Rewrite $\tilde{L}_1'\left(\Lambda^0 H - \tilde{L}\right)$ *as*

$$\begin{aligned}\tilde{L}_1'\left(\Lambda^0 H - \tilde{L}_1\right)/N &= \left(\tilde{L}_1 - \Lambda^0 H\right)'\left(\Lambda^0 H - \tilde{L}_1\right)/N + H\Lambda^{0\prime}\left(\Lambda^0 H - \tilde{L}_1\right)/N \\ &= O_p\left(\frac{1}{T^2}\right) + O_p\left(\frac{1}{T\sqrt{N}}\right) = O_p\left(\frac{1}{T\sqrt{N}}\right)\end{aligned}$$

from Lemma S.2 and Lemma S.3, respectively. The first term is

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} a_t^2 &= \frac{1}{T}\sum_{t=1}^{T}\left[\tilde{L}_1'\left(\Lambda^0 H - \tilde{L}_1\right)/N\right]^2 H^{-2} f_{0,t}^2 \\ &= \frac{1}{T}H^{-2}\left[\tilde{L}_1'\left(\Lambda^0 H - \tilde{L}_1\right)/N\right]^2 \sum_{t=1}^{T} f_{0,t}^2 \\ &= O_p\left(T^{-1}\right)O_p(1)\,O_p\left(\frac{1}{T^2 N}\right)O_p\left(T^2\right) = O_p\left(\frac{1}{TN}\right).\end{aligned}$$

The second term is

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} b_t^2 &= \frac{1}{N^2 T}\sum_{t=1}^{T}\left[\left(\tilde{L}_1' - H\Lambda^{0\prime}\right)e_t\right]^2 = \frac{1}{N^2 T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\left(\tilde{l}_{1,i} - H\lambda_{0,i}\right)e_{it}\right)^2 \\ &\leqslant \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N} e_{it}^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i} - H\lambda_{0,i}\right)^2\right) \\ &= \left(\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{l}_{1,i} - H\lambda_{0,i}\right)^2\right)\frac{1}{NT}\sum_{t=1}^{T}\sum_{i=1}^{N} e_{it}^2 = O_p\left(T^{-2}\right),\end{aligned}$$

from Lemma A.1 and $\frac{1}{NT}\sum_{t=1}^{T}\sum_{i=1}^{N} e_{it}^2 = O_p(1)$ *from the Assumption 4.3(4). The third term is*

$$\frac{1}{T}\sum_{t=1}^{T} c_t^2 = \frac{1}{TN^2}H^2\sum_{t=1}^{T}\left(\Lambda^{0\prime}e_t\right)^2 = \frac{1}{N}H^2\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N^{1/2}}\sum_{i=1}^{N}\lambda_{0,i}e_{it}\right)^2 = O_p\left(N^{-1}\right),$$

by Lemma 1(ii) in Bai and Ng (2002) which requries Assumption 4.2(1) and Assumption 4.3(3). Therefore, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(\tilde{y}_t - H^{-1}f_{0,t}\right)^2 = O_p\left(\frac{1}{T^2}\right) + O_p\left(\frac{1}{N}\right) = O_p\left(\delta_{NT}^{-2}\right),$$

where $\delta_{NT} = \min\left(\sqrt{N}, T\right)$.

B Finite Sample Performance of the Real-time PSY-factor Procedure

This section presents Monte Carlo simulation results for the real-time PSY-factor procedure. Instead of estimating the first factor from the entire sample, we compute the factor from a sample starting with the first available observation and ending with the current observation at τ using only historical information up to this point in time. Figures 1 and 2 report empirical sizes, SDRs and estimation accuracy of the bubble origination date by the real time recursive procedure. We set $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$. Results for the case of $\sigma_{00} = 0.02$ and $\sigma_{11} = 006$ were found to be similar and are not reported. The performance of the real-time procedure is almost identical to results in Section 5 of the main paper for the PSY-factor procedure.

Figure 1: Empirical sizes of the real-time PSY-factor procedure. The nominal size is 5%.

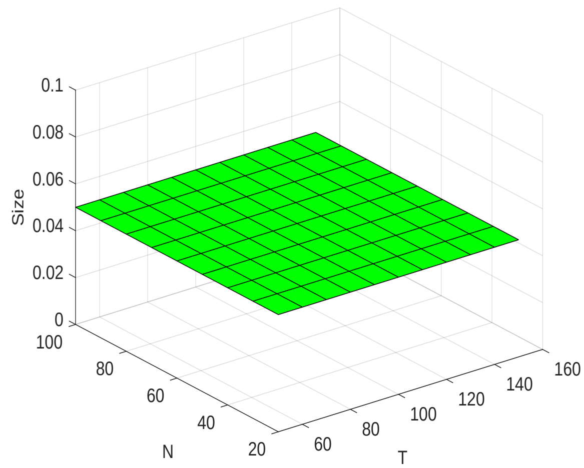
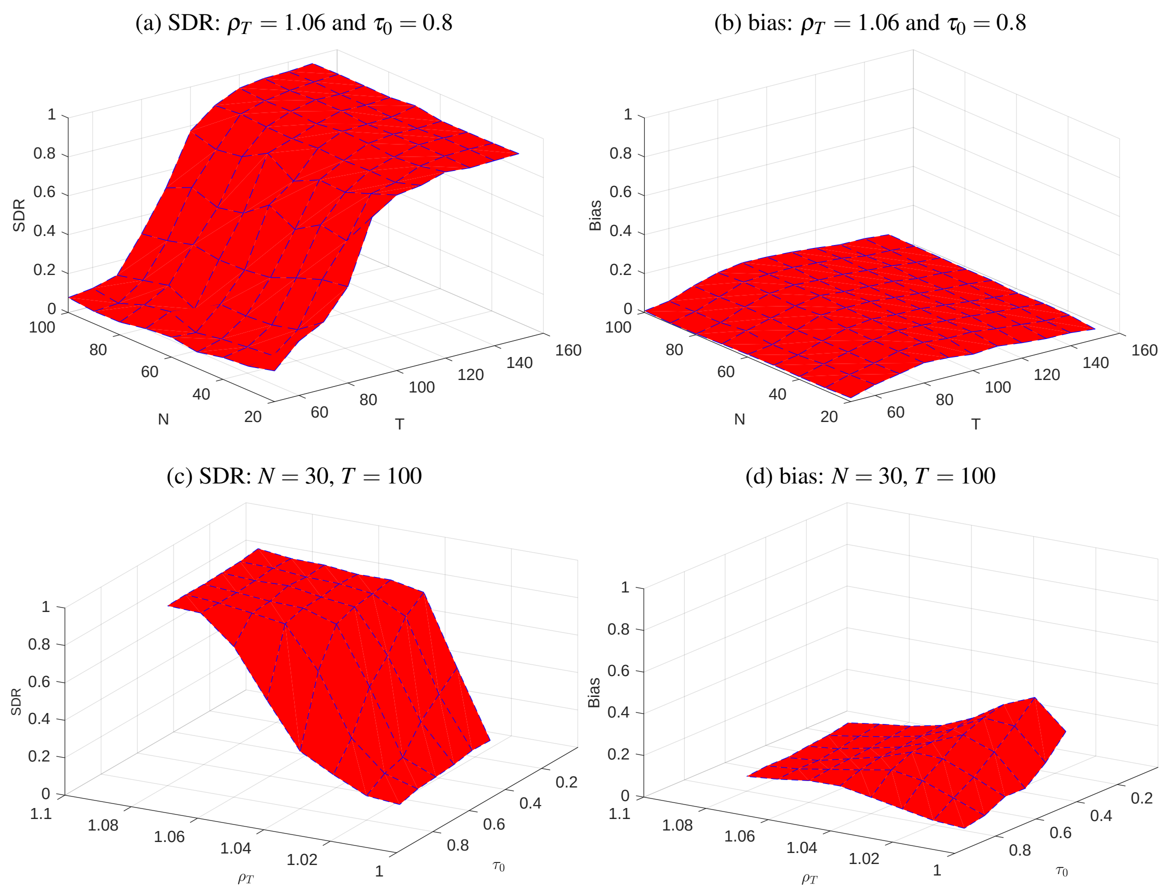


Figure 2: The successful detection rates (SDRs) and bias of the estimated bubble origination of the real time PSY-factor procedure. Parameter settings: $\tau_0 = 0.8$, $r_0 = 0.7$ and $\rho_T = 1.06$, $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$.



References

Bai, J. (2004), 'Estimating cross-section common stochastic trends in nonstationary panel data', *Journal of Econometrics* **122**(1), 137–183.

Chen, Y., Li, K. & Phillips, P. C. (2019), 'Mixed dynamic factor models—applied to explosive house prices', *Working Paper* .

Chen, Y., Phillips, P. C. & Shi, S. (2019), 'Common bubble detection in large dimensional financial systems'.