# Common Bubble Detection in Large Dimensional Financial Systems *

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**Abstract**

Price bubbles in multiple assets are sometimes nearly coincident in occurrence. Such near-coincidence is strongly suggestive of co-movement in the associated asset prices and likely driven by certain factors that are latent in the financial or economic system with common effects across several markets. Can we detect the presence of such common factors at the early stages of their emergence? To answer this question, we build a factor model that includes both I(1) and mildly explosive factors to capture normal and exuberant phases in such phenomena. The I(1) factor models the primary driving force of market fundamentals. The explosive factor models latent forces that underlie the formation of asset price bubbles, which typically exist only for subperiods of the sample. The paper provides an algorithm for testing the presence of and date-stamping the origination of price bubbles determined by latent factors in a large-dimensional system embodying many markets. Asymptotics of the bubble test statistic are given under the null of no common bubbles and the alternative of a common bubble across these markets. We prove consistency of a factor bubble detection process for the origination date of the commmon bubble. Simulations show good finite sample performance of the testing algorithm in terms of its successful detection rates. Our methods are applied to real estate markets covering 89 major cities in China over the period January 2003 to March 2013. Results suggest the presence of three common bubble episodes in what are known as China's Tier 1 and Tier 2 cities over the sample period. There appears to be little evidence of a common bubble in Tier 3 cities.

*Keywords:* Common Bubbles; Mildly Explosive Process; Factor Analysis; Date Stamping; Real Estate Markets.

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# 1 Introduction

Financial bubbles are conventionally defined as explosive deviations of asset prices from market fundamentals followed by a subsequent collapse (Blanchard 1979, Diba & Grossman 1988, Evans 1991). There is now considerable accumulated empirical evidence of bubbles in historical records of financial asset prices, including equity, commodity, and real estate markets.[1] In a large-dimensional financial system, bubbles may arise concurrently in many of the variables in the system. For instance, using univariate bubble testing methods Pavlidis et al. (2016) found evidence of bubble presence in 22 international housing markets between 1975 and 2013, observing high synchronization in three of the bubble episodes. In a similar way using a univariate bubble detection technique, Narayan et al. (2013) discovered abundant evidence of bubbles in 589 firms listed on the NYSE over the period from 1998 to 2008. The detected bubble episodes were observed to appear in clusters according to financial sector. Related work by Greenaway-McGrevy et al. (2019) found evidence supporting the presence of a common explosive factor in house prices for 16 cities in two countries (Australia and New Zealand) over the period 1986-2015.

The focus of the current paper is the econometric detection of a common factor underlying the presence of bubbles that appear in a large-dimensional financial system. While evidence of a potential common bubble factor appeared in the empirical work of Greenaway-McGrevy et al. (2019) such phenomena have not been analyzed in the factor modeling literature. In consequence, there are no formal tests, dating schemes, or asymptotic theory available for use in estimation and inference concerning bubble factor detection. A common bubble factor refers to the circumstance that the dynamics of asset prices within a financial system are dominated by a pervasive common explosive factor, in the sense that the number of nonzero loadings for the common explosive factor passes to infinity as the number of assets $N \to \infty$. This formulation allows for a finite number (or small infinity) of assets in the system to have zero loading on the explosive factor, so these assets are unaffected by the common bubble. The concept of a common bubble factor is related to the idea of co-explosiveness in autoregressive models (with either distinct or common explosive roots) that has been studied in Magdalinos & Phillips (2009), Chen et al. (2017), Nielsen (2010), Phillips & Magdalinos (2013). But unlike the concept of a common bubble factor, the number of variables in co-explosive systems is finite and all variables in these systems display explosive dynamics. The goal of the present paper is to provide econometric methods to test for the presence of a common bubble factor that may be determining dominant time series behavior in a large-dimensional system and to date-stamp the origination of this common bubble.

[1]Amongst a large and growing literature, see Phillips et al. (2011), Phillips & Yu (2011), Gutierrez (2012), Phillips & Yu (2013), Etienne et al. (2014*a*,*b*), Phillips et al. (2015*a*,*b*), Caspi et al. (2015), Adämmer & Bohl (2015), Figuerola-Ferretti et al. (2015), Pavlidis et al. (2016), Figuerola-Ferretti et al. (2016), Caspi (2016), Shi et al. (2016), Shi (2017), Greenaway-McGrevy & Phillips (2016), Hu & Oxley (2017*a*,*b*,*c*, 2018*a*,*b*), Phillips & Shi (2018, 2019), Milunovich et al. (2019).

The presence of asset price bubbles and potential commonality in bubble behavior across assets have important policy implications. Markets subject to common bubbles are extremely vulnerable to negative shocks and are exposed to the risk of system-wide failure, thereby entailing higher systemic risk (Brunnermeier & Oehmke 2013). In contrast, bubbles that occur independently in different markets without linkage or contamination seem likely to cause less system-wide damage. The procedures proposed in the present paper are intended to enable early identification of speculative behavior governed by a common latent factor that may expose financial markets to such system-wide risk. In addition, estimates of common explosive factors facilitate investigation of the underlying driving forces which produce this behavior and thereby offer potential guidance to governments and financial institution regulators in crafting policy to maintain economic and financial stability.

The identification of common bubble behavior also has important implications for the conduct of inference. Nielsen (2010) and Phillips & Magdalinos (2013) showed that maximum likelihood estimation of a vector autoregressive model is inconsistent when there are common explosive roots. Furthermore, the maximum likelihood estimator of co-explosive VAR models follows a mixed-normal limit distribution with Cauchy-type tail behavior rather than a normal distribution. To address the inconsistency, Phillips & Magdalinos (2013) propose an instrumental variable procedure for the consistent estimation of VAR models when the system contains co-explosive variables.

It is always possible to run univariate tests separately for bubble identification in each individual time series. But the presence of a common bubble characteristic across several time series, such as real estate prices in multiple regions or different metropolitan areas, is collective information of importance in understanding the phenomena and in assisting regulators to frame discretionary monetary policy. Cross section information from multiple time series is also necessary for identifying common bubbles. Furthermore, it is well known that the probability of making a false positive inference increases dramatically when univariate tests are applied repeatedly (in this case to a large number of assets), a phenomenon that is referred to as the multiplicity issue in the statistics literature.

The econometric procedure we propose here uses a factor model framework and involves two steps in the process of detecting a latent common bubble in the panel. In the first step we estimate the dominant common factor using a principal component (PC hereafter) approach. Factor estimation methods have been extensively used in applied economic research and asymptotic theory has been developed for stationary factor models in Bai & Ng (2002), Bai (2003), the I(1) factor model in Bai (2004), and most recently a mixed dynamic factor model with explosive, I(1), and stationary components in Chen, Li & Phillips (2019). The latter work is most relevant for the present study.

The second step in our procedure applies the recursive explosive root testing algorithm of Phillips, Shi and Yu (2015a&2015b, PSY hereafter) to the estimated dominant factor.

The PSY procedure is a commonly used bubble detection technique and has the capacity to consistently estimate bubble origination and termination dates (Phillips et al. 2015*b*). The test statistic used here to detect a common factor bubble and provide date-stamping is referred to as a PSY-factor testing algorithm. Under the null hypothesis that there is no common bubble, asset prices are assumed to be driven by an I(1) factor and an idiosyncratic error term. The limit distribution of the PSY-factor test statistic under this null is shown to be the same as that of the original PSY statistic, although the derivation of this result is complicated by the additional step of factor estimation.

The alternative hypothesis allows for the presence of a common bubble factor in a subsample of the panel. In this formulation the first part of the trajectory is governed by an I(1) factor, representing a period of market normalcy. The second part of the trajectory is driven by both an explosive factor and an I(1) factor, representing a period of market abnormality in relation to fundamentals or speculative market behavior. The estimated dominant first factor turns out to be a weighted average of the I(1), the explosive factor, and idiosyncratic errors, with weightings that depend on the estimated factor loadings. Under certain regularity conditions, we show that the PSY-factor test statistic diverges under the alternative. So the presence of a common speculative component in the data that leads to an explosive factor is identified and the procedure is shown to consistently estimate the origination date of the common bubble.

Simulations are used to investigate the empirical size, successful detection[2] rate, and the estimation accuracy of the common bubble origination date under various parameter settings. The results suggest satisfactory performance of the procedure in finite samples of the size typically used in empirical studies. As an illustration of the methodology, we apply the common bubble detection procedure to real estate markets of 89 cities in China over the time period 2003 to 2013. Three episodes of common explosive behavior in real estate prices are detected in 30 so-called Tier 1 and Tier 2 Chinese cities, whereas little evidence of a common bubble is found among the remaining 59 Tier 3 cities.

The rest of the paper is organized as follows. Section 2 describes the model specifications used for the null and alternative hypotheses. The econometric procedure for common bubble detection is introduced in Section 3. Section 4 provides the asymptotic properties of the test statistic under both the null and the alternative and shows the consistency of the estimated bubble origination date. Section 5 reports the results of the simulations investigating the finite sample performance of the procedure. The application to real estate markets in China is conducted in Section 6. Section 7 concludes. Proofs are collected in Appendices A, B, and C. Appendix D contains tables and figures.

[2]Successful detection occurs when the test indicates the presence of a common bubble and the estimated origination date occurs on or after the true origination date.

## 2 Model Specifications

A commonly used definition of bubble phenomena in financial markets is given by the present value identity (Diba and Grossman, 1988)

$$P_t = \sum_{s=0}^{\infty} \rho^s \mathbb{E}_t \left( R_{t+s} + U_{t+s} \right) + B_t, \tag{2.1}$$

where $P_t$ is the price of the asset, $R_t$ is the payoff received from the asset (i.e., rent for houses and dividends for stocks), $U_t$ is an unobservable market fundamental, and $\rho \in (0,1)$ is the discount factor. The bubble component $B_t$ satisfies the submartingale property

$$\mathbb{E}_t \left( B_{t+1} \right) = \frac{1}{\rho} B_t. \tag{2.2}$$

Asset prices are governed by the payoff and the unobservable variables in the absence of bubbles and hence are commonly believed to be at most I(1). Conversely, in the presence of bubbles, $B_t$ dominates the dynamics of asset prices and leads to explosive behavior of the data series.

We start the analysis with a simple model specification that differentiates normal and abnormal market behavior. In the absence of a common bubble factor, asset prices are assumed to be driven by an I(1) common factor and an idiosyncratic error, whereas in the presence of common speculative behavior prices are determined by an I(1) factor, a mildly explosive factor, and an idiosyncratic term. The mildly explosive factor allows for mild deviations from unit root I(1) behavior in the explosive direction and have been found useful in analyzing potentially explosive processes. Autoregressive models with such mildly explosive roots have been extensively studied and utilized in empirical research following Phillips & Magdalinos (2007*b*).

### 2.1 Under the Null: No Common Bubble

In this case with no common bubble, dynamics for the asset price processes $X_{it}$ are governed by market fundamentals so that

$$X_{it} = f_{0,t} \lambda_{0,i} + e_{it}, \tag{2.3}$$

where $f_{0,t}$ follows a unit root process

$$f_{0,t} = f_{0,t-1} + u_{0,t}. \tag{2.4}$$

The factor $f_{0,t}$ is assumed to capture the fundamental drivers of asset prices in normal market conditions subject to idiosyncratic errors $e_{it}$, which represent market variations.

In observation matrix form the model (2.3) can be rewritten as

$$X = F^0 \Lambda^{0\prime} + E, \tag{2.5}$$

where $X = (\underline{X}_1 \ldots, \underline{X}_N)$ is an $T \times N$ matrix of the observed data with $\underline{X}_i = (X_{i1}, \ldots, X_{iT})'$, $F^0 = (f_{0,1} \ldots, f_{0,T})'$ is a $T \times 1$ vector, $\Lambda^0 = (\lambda_{0,1} \ldots, \lambda_{0,N})'$ is an $N \times 1$ vector of loading coefficients, and $E = (\underline{e}_1 \ldots, \underline{e}_N)$ is an $T \times N$ matrix of idiosyncratic errors with $\underline{e}_i = (e_{i1}, \ldots, e_{iT})'$. At time $t$

$$X_t = \Lambda^0 f_{0,t} + e_t, \tag{2.6}$$

where $X_t = (X_{1t}, \ldots, X_{Nt})'$ and $e_t = (e_{1t}, \ldots, e_{Nt})'$. For each $i$, we have

$$\underline{X}_i = F^0 \lambda_{0,i} + \underline{e}_i.$$

## 2.2 Under the Alternative: Common Bubble Presence

Under the alternative hypothesis of a common bubble presence, asset prices are assumed to follow the factor dynamic mechanism

$$X_{it} = \begin{cases} f_{0,t}\lambda_{0,i} + e_{it} & \text{if } 1 \leqslant t \leqslant \lfloor \tau_0 T \rfloor \\ f_{1,t}\lambda_{2,i} + f_{0,t}\lambda_{0,i} + e_{it} & \text{if } \lfloor \tau_0 T \rfloor + 1 \leqslant t \leqslant T \end{cases}, \tag{2.7}$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The $\{f_{0,t}\}_1^T$ factor follows a unit root process as in (2.4), and the factor $\{f_{1,t}\}_{\lfloor \tau_0 T \rfloor + 1}^T$ is assumed to follow an autoregressive process with a mildly explosive root (Phillips & Magdalinos 2007*b*) such that

$$f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}, \tag{2.8}$$

where $\rho_T = 1 + \frac{c}{T^\eta}$ with rate parameter $\eta \in (0,1)$ and localizing coefficient $c > 0$. The initial value $f_{0,0}$ is assumed to be $O_p(1)$. The bubble factor $f_{1,t}$ is assumed to emerge at some period $T_r = \lfloor r_0 T \rfloor$ with $r_0 \in [0, \tau_0]$ and represents emergent positive sentiment about the market that translates into market exuberance when this sentiment enters into the price determination system at $T_0 + 1$ with $T_0 = \lfloor \tau_0 T \rfloor$. We assume that this market exuberance then lasts until the end of the sample period. Similar assumptions on the initiation of second regimes are commonly made in structural break models (e.g. Perron & Zhu (2005)).

The initial value $f_{1,T_r}$ of the bubble factor is assumed to be $O_p\left(T^{\eta/2}\right)$. It can easily be verified from the analysis in Phillips & Magdalinos (2007*b*) that the order of magnitude of the explosive factor at the break point $f_{1,T_0}$ is then $O_p\left(T^{\eta/2}\rho_T^{T_0 - T_r}\right)$, which reduces to $O_p\left(T^{\eta/2}\right)$ if the initial point coincides with the break date (i.e., $T_0 = T_r$). This setting of the initial value is similar to, but slightly less restrictive than, that of Phillips & Magdalinos (2007*b*), where the order of the initial value of the mildly explosive process is assumed to be $o_p\left(T^{\eta/2}\right)$.

The idiosyncratic errors $e_{it}$ in (2.7) may be serially correlated for each $i$. The factor specification error vector $u_t = (u_{0,t}, u_{1,t})'$ is taken to be $iid(0, \Sigma_u)$, in accordance with market efficiency in the first sample period, followed by market exuberance in the second sample period. Further details on the error conditions are given in the assumptions in Section 4, where broader conditions are discussed.

The data generating process under the alternative (2.7) can be rewritten as

$$X_{it} = \begin{cases} f_{1,t}\lambda_{1,i} + f_{0,t}\lambda_{0,i} + e_{it} & \text{if } t \in A \\ f_{1,t}\lambda_{2,i} + f_{0,t}\lambda_{0,i} + e_{it} & \text{if } t \in B \end{cases}, \tag{2.9}$$

where $\lambda_{1,i} = 0$ for $i = 1, \ldots, N$, $A = [1, T_0]$ and $B = [T_0 + 1, T]$ with $T_0 = \lfloor \tau_0 T \rfloor$. Let $F_1^0 = [f_{0,1}, \ldots, f_{0,T_0}]'$, $F_2^0 = [f_{0,T_0+1}, \ldots, f_{0,T}]'$, $F_1^1 = [f_{1,1}, \ldots, f_{1,T_0}]'$ and $F_2^1 = [f_{1,T_0+1}, \ldots, f_{1,T}]'$. Let $\Lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{1,N})' = \mathbf{0}_{N\times 1}$ and $\Lambda_2 = (\lambda_{2,1}, \ldots, \lambda_{2,N})'$.

We represent the model (2.9) in matrix form as

$$X = G\Gamma' + E, \tag{2.10}$$

where $G = [g_1, g_2, \ldots, g_T]'$ is a $T \times 3$ matrix, with

$$g_t' = [g_{1t}\ g_{2t}\ g_{3t}] = \begin{cases} [0, f_{1,t}, f_{0,t}], & \text{if } 1 \leqslant t \leqslant [\tau_0 T], \\ [f_{1,t}, 0, f_{0,t}], & \text{if } [\tau_0 T] + 1 \leqslant t \leqslant T \end{cases} \tag{2.11}$$

and $N \times 3$ matrix $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N]'$ with $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})'$ for $i = 1, \ldots, N$. The matrix $G$ can be rewritten as $G = [G_1, G_2, G_3]$ with

$$\begin{aligned} G_1 &= (g_{11}, \ldots, g_{1T})' = (0, \ldots 0, f_{1,T_0+1}, \ldots, f_{1,T})', \\ G_2 &= (g_{21}, \ldots, g_{2T})' = (f_{1,1}, \ldots, f_{1,T_0}, 0, \ldots 0)', \\ G_3 &= (g_{31}, \ldots, g_{3T})' = (f_{0,1}, \ldots, f_{0,T_0}, f_{0,T_0+1}, \ldots, f_{0,T})'. \end{aligned}$$

The explosive factor $f_{1,t}$ is split into components $G_1$ and $G_2$, while $G_3$ is the I(1) factor. The factor loading matrix is $\Gamma = [\Gamma_1\ \Gamma_2\ \Gamma_3]$ with $\Gamma_1 = \Lambda_2$, $\Gamma_2 = \Lambda_1$, and $\Gamma_3 = \Lambda_0$.

## 3 Econometrics of Common Bubble Identification

With the above model specification, the first factor is at most I(1) under the null of no common bubbles and is explosive in the presence of speculative behaviour. As such, detecting common bubbles is equivalent to distinguishing a martingale first factor from an explosive process. The proposed procedure consists of two steps. First, the leading common factor is estimated by principal components. In the second step we apply the PSY procedure to the estimated first factor to ascertain whether the leading factor manifests mildly explosive behavior.

### 3.1 Estimation of the First Common Factor

We estimate the first common factor using the following procedure. Assume the true number of factors is $r$ for the data $\{X_{it}\}$ with $i = 1, \dots, N$, and $t = 1, \dots, T$. Denote the common factors by the vector $\xi_t$ $(r \times 1)$ and the corresponding factor loadings by $l_i$ $(r \times 1)$. The objective function in the PC analysis is

$$\min_{\Xi, L} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( X_{it} - l_i' \xi_t \right)^2, \tag{3.1}$$

where $\Xi = (\xi_1, \dots, \xi_T)'$ and $L = (l_1, \dots, l_N)'$ is an $N \times r$ matrix. We impose a normalization condition on the loadings such that

$$\frac{1}{N} L' L = I_r. \tag{3.2}$$

The resulting solution for the factor loading, denoted by $\tilde{L}$, is $\sqrt{N}$ times the eigenvectors corresponding to the largest $r$ eigenvalues (denoted by $v$) of the $N \times N$ matrix $X'X$. The estimated $r$ factors, denoted by $\tilde{\Xi} = \left( \tilde{\xi}_1, \dots, \tilde{\xi}_T \right)'$, are

$$\tilde{\Xi} = X \tilde{L} \left( \tilde{L}' \tilde{L} \right)^{-1} = X \tilde{L} / N.$$

It is sufficient to obtain the first common factor for the purpose of bubble identification. For easy reference, we denote the estimated first common factor by $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_T)'$. Let $\tilde{L}_1$ be the estimated factor loading corresponding the first common factor. We then have

$$\tilde{y} = X \tilde{L}_1 / N.$$

### 3.2 The PSY Procedure

We apply the recursive evolving procedure of PSY to the estimated first common component $\tilde{y}_t$ to identify explosive behavior and characterize its nature, in particular to date-stamp the origination of any bubble that may be present. The regression model used for this purpose is

$$\Delta \tilde{y}_t = \alpha + \beta \tilde{y}_{t-1} + v_t, \tag{3.3}$$

where $v_t$ is the equation residual. The coefficient $\beta = 0$ under the null of no common bubble and $\beta > 0$ under the alternative. The centered OLS estimator of the autoregressive coefficient and its standard error are

$$\widehat{\beta} = \frac{T\sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1} - \sum_{t=1}^{T}\Delta\tilde{y}_t\sum_{t=1}^{T}\tilde{y}_{t-1}}{T\sum_{t=1}^{T}\tilde{y}_{t-1}^2 - \left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^2}, \tag{3.4}$$

$$s_{(\widehat{\beta})} = \widehat{\sigma}_{\tilde{y}}\left[\sum_{t=1}^{T}\tilde{y}_{t-1}^2 - \frac{1}{T}\left(\sum\tilde{y}_{t-1}\right)^2\right]^{-1/2}, \tag{3.5}$$

with $\widehat{\sigma}_{\tilde{y}}^2 = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\alpha} - \hat{\beta}\tilde{y}_{t-1}\right)^2$ and $\hat{\alpha} = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\beta}\tilde{y}_{t-1}\right)$. The standard unit root (Dickey-Fuller) t-statistic for the test of $H_0 : \beta = 0$ is

$$DF = \frac{\widehat{\beta}}{s_{(\widehat{\beta})}}. \tag{3.6}$$

To discuss the PSY mechanism, it is convenient to use fractional notation to represent observations within the sample. Suppose the observation of interest is $\tau$. To infer the presence of a common bubble characteristic at period $\tau$, PSY suggest applying regression (3.3) recursively to a group of structured subsamples. Let $\tau_{\min}$ be the minimum sample size required to initiate the regression. The starting date of the subsample regressions $\tau_1$ varies between 0 and $\tau - \tau_{\min}$, while the termination date $\tau_2$ of all subsamples is fixed on the observation of interest (i.e., $\tau_2 = \tau$). The DF statistics obtained from these subsample regressions are represented in the sequence $\{DF_{\tau_1,\tau_2}\}$. Inference concerning the presence of a common bubble is then based on the supremum of the DF statistic sequence, which is denoted $PSY_\tau$ and defined as

$$PSY_\tau = \sup_{\tau_1\in[0,\tau-\tau_{\min}],\tau_2=\tau}\left\{DF_{\tau_1,\tau_2}\right\}.$$

Let $\beta_T$ be the significance level and $cv_{\beta_T}$ be the $100(1-\beta_T)\%$ critical value of the test. If a common bubble is detected, then its origination date, $\hat{\tau}_0$, is identified to be the first chronological observation where the test statistic sequence exceeds the critical value. That is,

$$\hat{\tau}_0 = \inf_{\tau\in[\tau_{\min},1]}\left\{\tau : PSY_\tau > cv_{\beta_T}\right\}.$$

# 4 Asymptotics

We start by stating assumptions on the common factors, loadings, and errors which assist in the development of the asymptotic theory. Throughout, the notation $M$ is used to denote a (possibly large) constant whose value may change in each location, and $\Longrightarrow$ signifies weak convergence on the relevant probability space. We assume that $N$ and $T$ pass to infinity at the same rate, so that $N/T \to c$ for some constant $c > 0$.

## 4.1 Model Assumptions

**Assumption 4.1** ***(Common factors):*** *Define the filtration* $\mathcal{F}_t = \sigma\{u_t, u_{t-1}, ...\}$ *where* $u_t = (u_{0,t}, u_{1,t})'$, *and let* $\{u_t, \mathcal{F}_t\}$ *be a martingale difference sequence (mds) with* $\mathbb{E}(u_t u_t' | \mathcal{F}_{t-1}) = \Sigma_u$, *and*

$$\Sigma_u = \begin{bmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{bmatrix} > 0$$

*and* $\sup_t \mathbb{E} \|u_t\|^{2+\varsigma} \leqslant M$ *for some* $\varsigma > 0$ *and for all* $t \leqslant T$.

**Assumption 4.2** ***(Factor loadings):***

*(1) Under the null of no bubble factor, as in (2.3), deterministic loadings* $\{\lambda_{0,i}\}$ *are assumed to satisfy* $\|\lambda_{0,i}\| \leqslant M$ *and stochastic loadings to satisfy* $\sup_i \mathbb{E} \|\lambda_{0,i}\|^4 \leqslant M$, *both with* $\Lambda^{0\prime}\Lambda^0/N \to_p \Sigma_\Lambda$ *as* $N \to \infty$ *where* $\Sigma_\Lambda > 0$ *is nonrandom.*

*(2) Under the alternative of a bubble factor, as assumed in model (2.10), deterministic loadings* $\{\gamma_i\}$ *are assumed to satisfy* $\|\gamma_{ii}\| \leqslant M$, *stochastic loadings to satisfy* $\sup_i \mathbb{E} \|\gamma_i\|^4 \leqslant M$, *and the loading moment matrix*

$$\Gamma'\Gamma/N = \frac{1}{N} \begin{bmatrix} \Lambda_2'\Lambda_2 & \cdot & \cdot \\ \Lambda_2'\Lambda_1 & \Lambda_1'\Lambda_1 & \cdot \\ \Lambda_2'\Lambda_0 & \Lambda_1'\Lambda_0 & \Lambda_0'\Lambda_0 \end{bmatrix} \to_p \Pi := \begin{bmatrix} \Pi_{22} & \cdot & \cdot \\ 0 & 0 & \cdot \\ \Pi_{20} & 0 & \Pi_{00} \end{bmatrix},$$

*which is positive-definite. Here,* $\frac{1}{N}\Lambda_k'\Lambda_l = \frac{1}{N}\sum_{i=1}^N \Lambda_{ik}\Lambda_{il} \to \Pi_{kl}$ *as* $N \to \infty$.

**Assumption 4.3** ***(Time and cross-section dependence and heteroskedasticity)****: For some number* $M < \infty$,

*(1)* $\mathbb{E}(e_{it}) = 0$, $\sup_{i,t} \mathbb{E}|e_{it}|^8 \leqslant M$;

*(2)* $\mathbb{E}(\underline{e}_i'\underline{e}_j/T) = \mathbb{E}\left(\frac{1}{T}\sum_{t=1}^T e_{it}e_{jt}\right) = \gamma_T(i,j)$ *with* $\sup_{T\geqslant 1}\sup_{i,j}|\gamma_T(i,j)| \leqslant M$, *and*

$$\sup_{N\geqslant 1} \frac{1}{N} \sum_{i,j=1}^N |\gamma_T(i,j)| \leqslant M;$$

*(3)* $\mathbb{E}(e_{it}e_{jt}) = \tau_{ij,t}$ *with* $|\tau_{ij,t}| \leq |\tau_{ij}|$ *for some* $\tau_{ij}$ *and for all* $t$, *and* $\frac{1}{N}\sum_{i,j=1}^N |\tau_{ij}| \leqslant M$;

*(4)* $\mathbb{E}(e_{it}e_{js}) = \tau_{ij,ts}$ *and* $\sup_{N\geqslant 1, T\geqslant 1} \frac{1}{NT}\sum_{i,j=1}^N \sum_{s,t=1}^T |\tau_{ij,ts}| \leqslant M$;

*(5) For every* $(i,j)$, $\sup_{T\geqslant 1} \mathbb{E}\left|T^{-1/2}\sum_{t=1}^T [e_{it}e_{jt} - \mathbb{E}(e_{it}e_{jt})]\right|^4 \leqslant M$.

**Assumption 4.4** $\sup_{i\geqslant 1} |\frac{1}{T}\sum_{t=1}^T f_{0,t-1}e_{it}| = O_p(1)$ *as* $T \to \infty$.

**Assumption 4.5** $\{\lambda_i\}, \{u_t\}$, *and* $\{e_{it}\}$ *are mutually independent.*

Assumption 4.1 concerns the common factor errors $u_t = \{u_{0,t}, u_{1,t}\}$ which are assumed to be *mds* with uniform $2 + \zeta$ moments. This condition is convenient, treating the component errors $\{u_{0,t}, u_{1,t}\}$ in the two periods commonly. It may be relaxed to allow (i) *mds* errors $\{u_{0,t}\}$ during the efficient market period, and (ii) more general weak dependence for $\{u_{1,t}\}$ during the explosive period, as in Phillips & Magdalinos (2007*a*), Magdalinos & Phillips (2009). No distributional assumptions are needed and the uniform moment condition is weak, so the methods proposed can be applied widely in empirical work, including to financial market data.

Assumption 4.2 concerns the loading coefficients, whose moment matrices $\Lambda^{0\prime}\Lambda^0/N$, $\Gamma'\Gamma/N$ are assumed to converge to positive definite matrices as $N \to \infty$, a condition which helps to ensure identifiability of the factor structures. So, if a factor had only a finite number of nonzero loadings, it would not be treated as a common factor in our framework but would instead be absorbed within the idiosyncratic errors $e_{it}$.

Assumption 4.3 allows for time and cross sectional dependence and conditional heteroskedasticity, as in Bai (2004). Assumption 4.4 requires the uniform boundness over $i$ of the time series sample covariances between $f_{0,t-1}$ and $e_{it}$ and is stronger than simply requiring weak convergence of such sample covariances for all $i$ as in Bai (2004). The independence between $u_t$ and $e_{it}$ in Assumption 4.5 eliminates endogeneity in our framework, just as in the cointegrated factor model of Bai (2004). Under the null hypothesis, the situation is analogous to that of Bai (2004) with an integrated factor. In such cases, the model can be rewritten as a dynamic factor model by projection of $e_{it}$ on $u_t$ and suitably augmenting the regression equation, leading to a dynamic factor model as discussed in Bai (2004).[3] However, in our case under the alternative, the presence of a mildly explosive factor accommodates dependence between $u_t$, and $e_{it}$ as shown in the cointegrating regression analysis of Magdalinos & Phillips (2009) with mildly explosive regressors. We therefore expect that the procedures for identifying and estimating the explosive factor in our framework retain validity under endogeneity, although formal analysis of this extension is not pursued in the present paper and left for subsequent work.

Additional assumptions used in the general setting of Bai (2004) are not required in the present paper. This is because in the model structure employed here there is no need to estimate the number of factors or to show uniform consistency of the estimated first factor.

## 4.2 Asymptotics Under the Null Hypothesis

The following Lemma shows consistency of the estimated first factor. This result is useful in developing an asymptotic theory of inference for quantities that relate to this estimated factor

[3] In other work that does not involve explosive or nonstationary processes, Pesaran (2006) allows for endogeneity between the factor and the residuals by using cross section averages in a multifactor regression model.

$\tilde{y}_t$. In particular, the theory is employed in deriving asymptotics for the bubble identification procedure.

**Lemma 4.1** *Under the data generating process* (2.3) *and Assumptions 4.1-4.5, we have*

$$\delta_{NT}^2 \left( \frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_t - H^0 f_{0,t} \right|^2 \right) = O_p(1) \tag{4.1}$$

*where* $\delta_{NT} = \min\left(\sqrt{N}, T\right)$ *and* $H^0 = \lambda_{NT} \left(\frac{F^{0\prime}F^0}{T^2}\right)^{-1} \left(\frac{\Lambda^{0\prime}\tilde{L}_1}{N}\right)^{-1}$ *with* $\lambda_{NT}$ *being the largest eigenvalue of* $\frac{1}{NT^2}X'X$.

Lemma 4.1 reveals that the first factor can be identified up to a transformation given by $H^0$. The proof of 4.1 follows directly as in Bai (2004) and Chen, Li & Phillips (2019) and is given for convenience in the Online Supplement (Chen, Phillips & Shi 2019). While Bai (2004) shows consistency of factor estimates in the presence of $I(1)$ factors (and uniform consistency under stronger moment conditions) subject to a normalization condition for the factors of the form $\Xi'\Xi/T^2 = I_r$, Chen, Li & Phillips (2019) provide consistency results under a factor model specification that includes an explosive factor as well as I(1) and stationary factors.

Next, we develop asymptotics for a standard unit root test constructed from the first estimated factor, $\tilde{y}_t$, under the null (2.3).

**Theorem 4.2** *Under the null specification (2.3) and Assumptions 4.1, 4.2(1), 4.3, 4.4, and 4.5, as* $N, T \to \infty$,

$$DF \Rightarrow \frac{\int_0^1 W(r)\, dW(r) - W(1) \int_0^1 W(r)\, dr}{\left[ \int_0^1 W(r)^2 dr - \left[ \int_0^1 W(r)\, dr \right]^2 \right]^{1/2}},$$

*where* $W(\cdot)$ *denotes the standard Brownian motion.*

Derivation of the asymptotic behavior of the DF statistic (3.6) follows standard lines. Although complicated by the fact that the test relies on the estimated factor, the derivation proceeds as usual because the fast convergence of $\tilde{y}_t$ to $H^0 f_{0,t}$ ensures that the limit distribution is unaffected by factor estimation and is identical to that of the DF statistic computed from the original data (as in, e.g., Phillips (1987)). An outline of the derivations is provided in Appendix A. With this result in hand, the limit behavior of the PSY test applied to the fitted factor also follows in the standard manner (Phillips et al. 2015*a*,*b*).

**Theorem 4.3** *Under the null specification of model (2.3) and Assumption 4.1, 4.2(1), 4.3,* *4.4, and 4.5, as $N, T \to \infty$,*

$$PSY_\tau \Rightarrow \sup_{\tau_1 \in [0, \tau - \tau_{\min}], \tau_2 = \tau} \left\{ \frac{\tau_w \int_{\tau_1}^{\tau_2} W(r)\, dW(r) - [W(\tau_2) - W(\tau_1)] \int_{\tau_1}^{\tau_2} W(r)\, dr}{\tau_w^{1/2} \left[ \int_{\tau_1}^{\tau_2} W(r)^2\, dr - \left( \int_{\tau_1}^{\tau_2} W(r)\, dr \right)^2 \right]^{1/2}} \right\}, \quad (4.2)$$

*where $\tau_w = \tau_2 - \tau_1$.*

The proof applies functional limit theory of the component elements of the statistic under the null and a version of continuous mapping applied to certain indexed functionals of these elements, just as in theorem 1 of Phillips et al. (2015*a*). The limit result (4.2) for the PSY-factor test statistic is then identical to that of the original PSY statistic (i.e., $F_{r_2}(W, r_0)$ in Phillips et al. (2015*a*)). The details of the proof are omitted for brevity.

### 4.3 Asymptotics Under the Alternative

We start with a useful representation of the first common factor under the alternative.

**Lemma 4.4** *Under the alternative (2.7) and Assumptions 4.1, 4.2(2), 4.3, 4.4, and 4.5, the estimated first common factor has the form*

$$\tilde{y}_t = a_{N,T} f_{1,t} + b_{N,T} f_{0,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{l}_{i1} e_{it}, \quad (4.3)$$

*where the $\tilde{l}_{i1}$ are the estimated loadings of the first factor, $a_{N,T} = \frac{1}{N}\sum_{i=1}^{N} \tilde{l}_{i1}\lambda_{2,i}$ and $b_{N,T} = \frac{1}{N}\sum_{i=1}^{N} \tilde{l}_{i1}\lambda_{0,i}$. Further, $a_{N,T} = O_p(1)$, $b_{N,T} = O_p(1)$, $\frac{1}{N}\sum_{i=1}^{N} \tilde{l}_{i1} = O_p(1)$, and $\frac{1}{N}\sum_{i=1}^{N} \tilde{l}_{i1} e_{it} = O_p(1)$.*

Under the alternative (2.7), the estimated first factor is therefore a weighted average of the two factors $f_{0,t}$ and $f_{1,t}$ and the idiosyncratic errors. The weights depend on the estimated factor loadings $\tilde{l}_{i1}$ and the true loading coefficients $\lambda_{0,i}$ and $\lambda_{2,i}$ and are shown to have order $O_p(1)$. In what follows, the subscripts $N, T$ will be omitted from the weights $\{a_{N,T}, b_{N,T}\}$ to avoid notational clutter.

Next, we derive the asymptotic properties of $\hat{\beta}$ and the unit root statistic $DF$ based on quantities presented in Lemma C.1 of Appendix C.

**Theorem 4.5** *Under the alternative (2.7) the following asymptotics hold as $N, T \to \infty$:*

$$\hat{\beta} = \frac{c}{T^\eta}[1 + o_p(1)]; \quad (4.4)$$

*and*

$$DF = \frac{\widehat{\beta}}{s.e\left(\widehat{\beta}\right)} = O_p\left(T^{1-\eta}\right). \tag{4.5}$$

According to Theorem 4.5, $\hat{\beta}$ can be regarded as consistent for the deviation $\rho_T - 1$ in (2.8) under the local alternative. The order of magnitude of the DF statistic depends asymptotically on the power parameter $\eta \in (0,1)$ that defines the magnitude of this local alternative and thereby the explosive strength of the autoregressive coefficient $\rho_T = 1 + cT^{-\eta}$, which rises as $\eta$ decreases. Correspondingly, the DF statistic diverges to positive infinity at the rate $O_p\left(T^{1-\eta}\right)$, where the rate increases with the explosive strength.[4]

Under the alternative of model (2.7), the sample period has two regimes – A and B as defined in (2.9). There are potentially three types of subsample regressions for the PSY procedure: $\tau_1 \in A$ and $\tau_2 \in B$ (Case 1), $\tau_1, \tau_2 \in A$ (Case 2), and $\tau_1, \tau_2 \in B$ (Case 3). From Theorem 4.2, the order of magnitude of $DF_{\tau_1,\tau_2}$ for Case 2 is $O_p(1)$. For both Case 1 and 3, the explosive factor $f_{1,t}$ dominates the I(1) component and hence the orders of magnitude of $DF_{\tau_1,\tau_1}$ under these scenarios are identical to those in (4.5). Consequently,

$$PSY_\tau = \begin{cases} O_p(1) & \text{if } \tau \in N \\ O_p\left(T^{1-\eta}\right) & \text{if } \tau \in B \end{cases}.$$

We are now able to deduce the asymptotic behavior of the bubble originating date estimator $\hat{\tau}_0$ in the factor model (2.7).

**Theorem 4.6** *Under the alternative (2.7), $\hat{\tau}_0 \to \tau_0$ if the divergence rate of the PSY critical value $cv_{\beta_T} \to \infty$ falls between $O_p(1)$ and $O_p(T^{1-\eta})$, i.e.,*

$$\frac{1}{cv_{\beta_T}} + \frac{cv_{\beta_T}}{T^{1-\eta}} \to 0.$$

Theorem 4.6 provides rate conditions on the localizing power cofficient $\eta$ under which the bubble origination date may be consistently estimated. The proof follows directly from Phillips et al. (2015*b*) and is therefore omitted.

## 5 Simulations

This section explores the finite sample performance of the common bubble detection procedure. Empirical sizes under the null and successful detection rates (SDR) are reported along

[4]The order of magnitude of the DF statistic differs slightly from that of Phillips et al. (2015*b*), which is $O_p\left(T^{1-\eta/2}\right)$. This difference arises from the distinct assumptions regarding the initialization of the explosive regime/factor. Here it is assumed that $f_{1,T_r} = O_p(T^{\eta/2})$, whereas the explosive regime of Phillips et al. (2015*b*) is assumed (implicitly) to start from a value of $O_p(T^{1/2})$.

with the average bias of the estimated bubble origination date under the alternative. The number of replications is 2,000 in all simulations.

### Multiplicity Issue

Critical values are obtained by simulation. The data generating process is (2.3)-(2.4). Let $\tilde{y}_t^s$ be the estimated first common factor detected in simulation $s$, where $s = 1, \cdots, 2000$. The PSY test statistic sequence $\{PSY_t^s\}_{t=T_0}^T$ is then computed from the time series $\tilde{y}_t^s$. In order to control in this recursive testing procedure for multiplicity[5] we calculate the maximum value of this test statistic sequence

$$\mathcal{M}_t^s = \max_{t \in [T_0, T]} \left( PSY_t^s \right)$$

and take the 95% percentiles of the $\{\mathcal{M}_t^s\}_{s=1}^{2000}$ sequence as the critical value of the PSY-factor procedure. By doing so, we maintain a probability of 5% for making at least one false positive conclusion over the entire sample period.

### Data Generating Processes

The data generating process is (2.3)-(2.4) under the null and (2.7), (2.4), and (2.8) under the alternative. The factor loadings $\lambda_{0,i}$ and $\lambda_{1,i}$ are drawn randomly from a uniform distribution between 0 and 2. The standard deviation of the idiosyncratic error $\sigma_e$ is set to 0.1. These parameter settings are compatible with our later empirical application to Chinese housing markets. The estimated loadings of both factors range between 0.3 and 1.7, while the estimated standard deviation of the idiosyncratic error term (assuming two factors) is around 0.1 for both Group I (Tier 1 and 2 cities) and Group II (Tier 3 cities). Note that under the null hypothesis the parameter settings of $f_{0,0}$, $\sigma_e$, and $\sigma_{00}$ do not affect the distribution of the ADF statistic, consistent with the results of Theorem 4.2.

Under the alternative, we allow the explosive rate $\rho_T$ to vary between 1.01 and 1.08 (with increments of 0.01). We consider two combinations of $\sigma_{00}$ and $\sigma_{11}$: $\{\sigma_{00} = 0.08, \sigma_{11} = 0.1\}$ and $\{\sigma_{00} = 0.02, \sigma_{11} = 0.06\}$. The first parameter setting corresponds to Group I in the empirical application and the second one is for Group II. Specifically, we calibrate the $f_{0,t}$ process to the normal periods in the estimated first factor (from January 2011 onwards for Group I and from February 2005 onwards for Group II). The estimated values for $\sigma_{00}$ are 0.08 and 0.02 for Groups I and II, respectively. Similarly, we calibrate the $f_{1,t}$ process to the fast expansion periods in Group I (May 2009 to December 2010) and Group II (January 2003 to January 2005). The selections of the sample periods for $f_{0,t}$ and $f_{1,t}$ are guided by

[5] The probability of making a Type I error rises with the number of hypotheses in a recursive test sequence, which is referred to as the multiplicity issue in testing. This tendency towards oversizing may be controlled by using a familywise critical value. See PSY for discussion and for the development of a bootstrap procedure which assists in controlling size in such cases.

the empirical results. We estimate (2.8) by the indirect inference approach to reduce biases as in Phillips et al. (2011). The estimated values of $\sigma_{11}$ are 0.1 and 0.06 for Groups I and II, respectively, and the corresponding estimated $\rho_T$ values are 1.08 and 1.06.

The initial values of the I(1) and explosive factors are set to unity (i.e., $f_{0,0} = 1$ and $f_{1,T_r} = 1$). To avoid sudden dramatic jumps at the break point $T_0 + 1$, we subtract the simulated $f_{1,t}$ for $t \in [T_r, T]$ by the value of $f_{1,T_0}$ so that the explosive factor takes value zero at period $T_0$.

Figure 1: Typical realizations of the data generating process under the alternative. Parameter settings are: $f_{0,0} = f_{1,T_r} = 1, \sigma_e = 0.1, r_0 = 0.7, \tau_0 = 0.8, \rho = 1.06, N = 30$ and $T = 60$. The vertical lines indicate the starting date of the common bubble episode.

(a) $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$ (b) $\sigma_{00} = 0.02$ and $\sigma_{11} = 0.06$

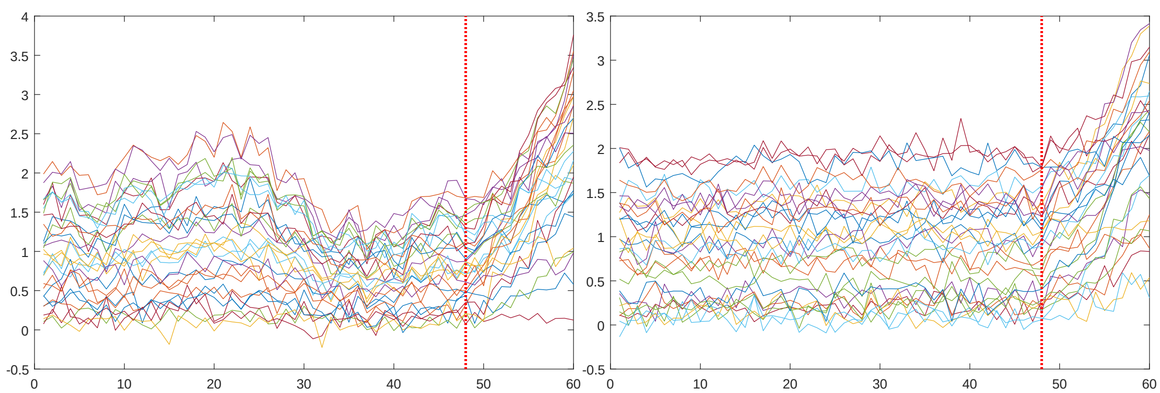


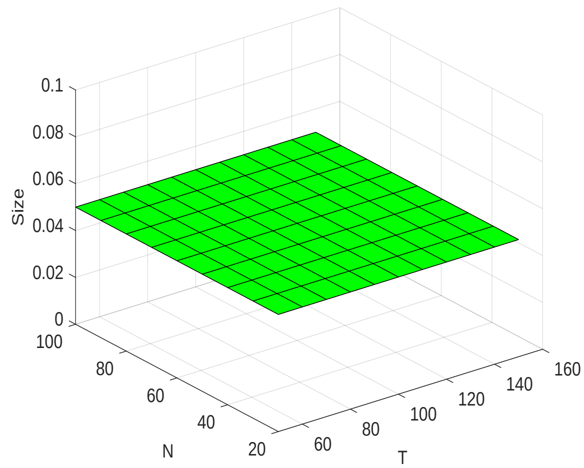

Figure 1 displays two typical realizations of the data generating process under the specified alternatives, with $\{\sigma_{00} = 0.08, \sigma_{11} = 0.1\}$ in panel (a) and $\{\sigma_{00} = 0.02, \sigma_{11} = 0.06\}$ in panel (b). The volatility during normal market conditions in panel (a) is clearly larger than that of panel (b) as expected from the parameter settings.

## Finite Sample Performance

In calculating test size, various $(N, T)$ settings are employed. The time series sample size $T$ takes values from 40 to 150 in increments of 10 and the number of assets $N$ runs from 10 to 100 again in increments of 10. Figure 2 displays the empirical test sizes of the PSY-factor procedure for these $(N, T)$ settings and the results show that empirical size is generally close to the nominal 5% size in all cases.

The successful detection rate of the common bubble detection procedure and bias in the estimated bubble origination date (i.e., $\frac{1}{2000}\sum_{s=1}^{2000} \hat{\tau}_0^s - \tau_0$) are illustrated in Figures 3-4. Successful detection rates are reported in the first column and biases in the second column. In Figure 3, we fix the break point $\tau_0$, the initialization point of $f_{1t}$, and the autoregressive coefficient of the explosive factor (i.e., $\tau_0 = 0.6$, $r_0 = 0.5$ and $\rho_T = 1.06$) and allow $N$ and $T$

Figure 2: Empirical size of the PSY-factor procedure with nominal size 5%.



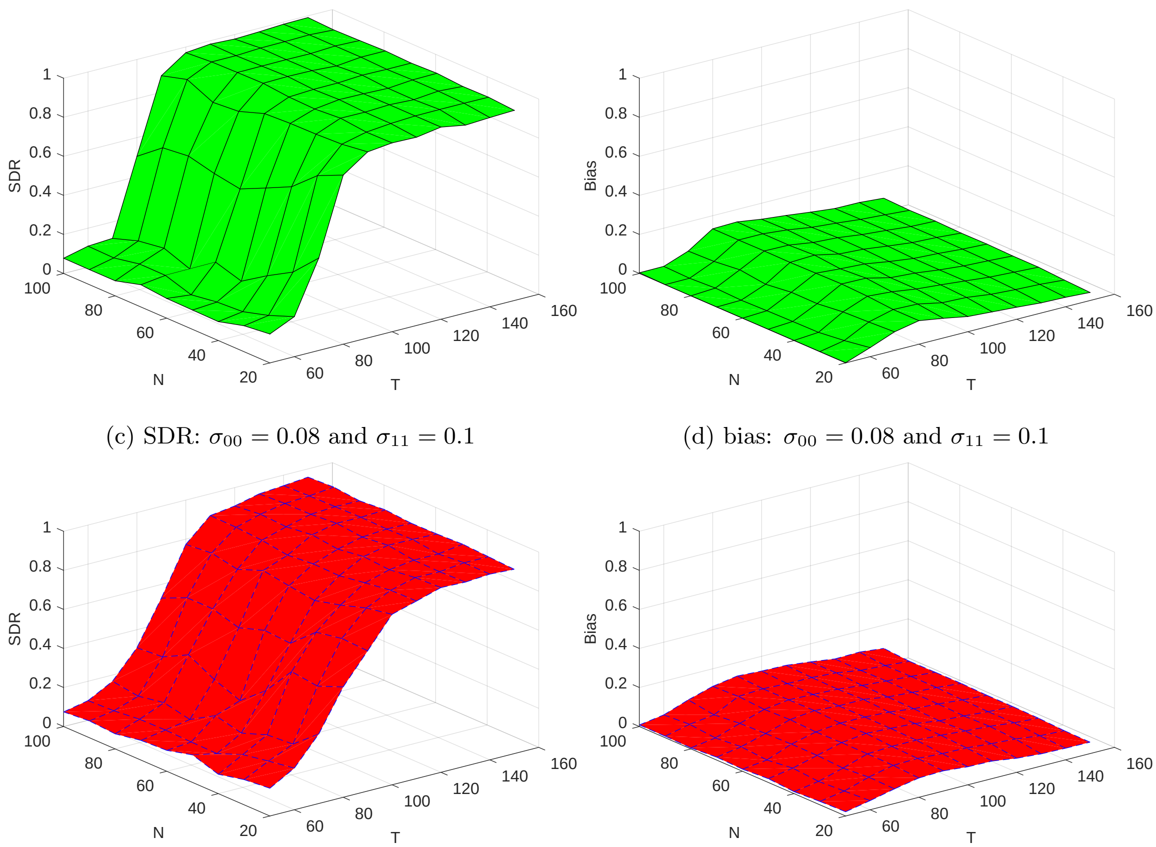

to vary as in Figure 2. Panels (a) and (b) are for the case of $\sigma_{00} = 0.02$ and $\sigma_{11} = 0.06$ and panels (c) and (d) are obtained from the setting of $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$. Note that the ratio between the volatilities of the explosive factor and the I(1) factor (i.e., $\sigma_{11}/\sigma_{00}$) is 3 in the former and 1.25 for the latter.

The following comments are in order. First, the SDR and bias of the PSY-factor procedure vary little with the number of assets $N$. Second, as the time span $T$ lengthens, while the bias of $\hat{\tau}_0$ remains roughly the same, the SDR increases dramatically. Additional time dimension information therefore lends considerable assistance in identifying explosive dynamics. Third, comparing panels (a) and (c), it is evident that the detection rate is higher in panel (a). It is therefore easier for the procedure to detect transition to an explosive period when the gap in the volatilities between the regimes is larger (i.e., in the case of $\sigma_{00} = 0.02$ and $\sigma_{11} = 0.06$).

In Figure 4, we fix $N$ and $T$, but allow $\tau_0$ and $\rho_T$ to take various values. The other parameters are $N = 30$, $T = 100$, $r_0 = \tau_0 - 0.1$, $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$.[6] The explosive rate $\rho_T$ varies between 1.01 and 1.08 (in increments of 0.01) and the break point $\tau_0$ ranges from 0.3 to 0.8 (with increments of 0.1). As expected, it is much easier to detect episodes that expand at a greater rate (i.e., when $\rho_T$ is further above unity). From panel (a), we see that as $\rho_T$ becomes larger, the SDR rises rapidly and the bias of the estimated origination date reduces. Furthermore, the SDR of the test declines slightly when $\tau_0$ increases. It is therefore harder for the procedure to detect a common bubble when it occurs later in the sample period. This is because there are fewer observations from the explosive dynamic episode available for the test. We observe some slight decrease in the bias as $\tau_0$ increases. This decrease occurs by construction as the estimated origination dates of successful detected episodes fall between $\tau_0$ and 1 and the bias can only vary between 0 and $1 - \tau_0$. Thus, as $\tau_0$ moves closer to unity the bias inevitably becomes smaller.

[6]Unreported simulations show that the SDR is higher when the $f_{1,t}$ process initiates at an ealier stage (i.e., when $r_0$ is smaller).

Figure 3: The successful detection rates and bias of the estimated bubble origination. Parameter settings: $\tau_0 = 0.8$, $r_0 = 0.7$ and $\rho_T = 1.06$.




(a) SDR: $\sigma_{00} = 0.02$ and $\sigma_{11} = 0.06$

(b) bias: $\sigma_{00} = 0.02$ and $\sigma_{11} = 0.06$

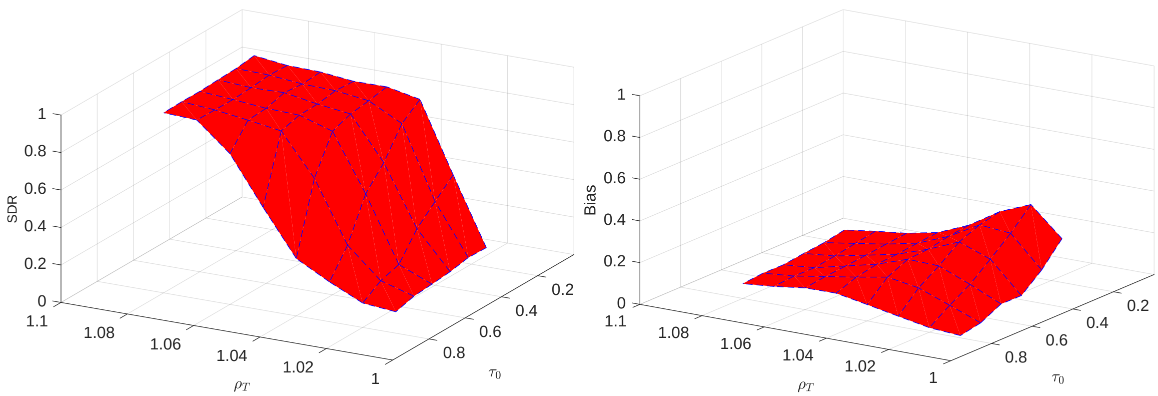
(c) SDR: $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$

(d) bias: $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$

Next, we consider a real-time implementation of the PSY-factor procedure. Specifically, instead of estimating the first factor from the entire sample, for each observation of interest $\tau$ we compute the factor from a sample starting with the first available observation and ending with the current observation at $\tau$ using only historical information up to this point in time. The empirical sizes, SDRs and estimation accuracy of the bubble origination dates by the recursive procedure are presented in Figure 1 and 2 in the Online Supplment (Chen, Phillips & Shi 2019). No major differences between the finite sample performance of the PSY-factor procedure and this real-time implementation are observed.

Figure 4: The successful detection rates and bias of the estimated bubble origination. Parameter settings: $N = 30$, $T = 100$, $r_0 = \tau_0 - 0.1$, $\sigma_{00} = 0.08$ and $\sigma_{11} = 0.1$.

(a) SDR (b) bias




# 6 Empirical Application: China Real Estate Markets

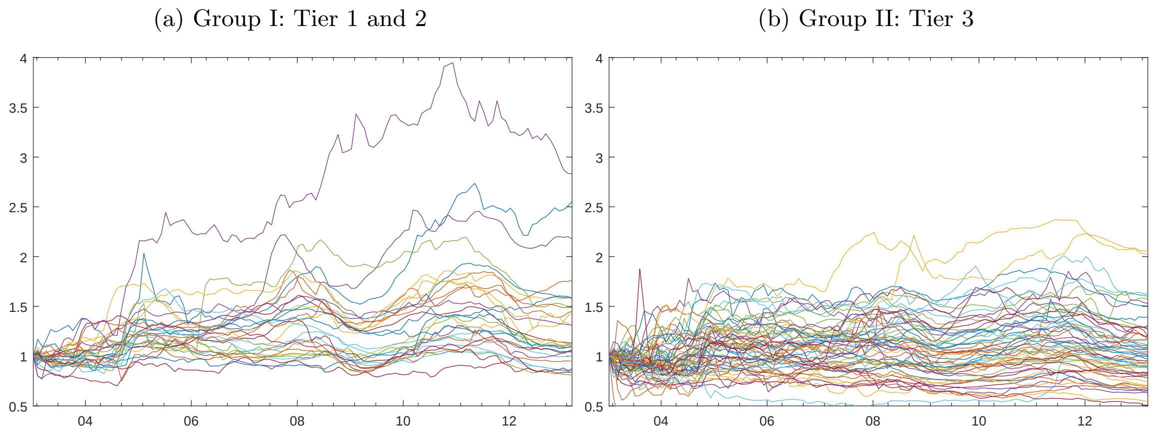
## 6.1 Data Description

We study housing markets in 89 major Chinese cities.[7] The sample includes 4 Tier 1, 26 Tier 2, and 59 Tier 3 cities. A list of these cities is given in Table 1. Monthly house prices are compiled by Fang et al. (2016), based on sequential sales of new homes within the same housing development. The longest available period contains 123 observations, running from January 2003 to March 2013. Underlying market fundamentals are proxied by urban disposable income per capita, which measures per capita income received by urban residents within each of the cities. The data are obtained from China City Yearbook, normalized to unity at the beginning of the sample period.

We split the sample into two groups. The first group includes all Tier 1 and 2 cities (group I), while the second group contains Tier 3 cities (group II). Figure 5 presents the housing price-to-income ratios (PIR) of group I (left panel) and group II (right panel). The variation within group I is larger than group II. We observe a dramatic increase of the price-to-income ratio in group I around 2007-2008 and again during 2010-2011. The 2007-2008 episode was led by cities Wenzhen, Shenzhen and Ningbo. The rise during 2010-2011 is of a larger magnitude. The price-to-income ratio reaches 3.95 in Wenzhou in December 2010 and 2.74 in Beijing in early 2011, followed by Shenzhen (2.5) and Ningbo (2.2) in 2010. The most outstanding cities within group II are Baoding and Ningde, especially after 2007. Figure 7 displays the average PIR over the sample period for each city. Similar to what we observed from Figure 5, the average PIR of Wenzhou is the highest and is well above the national average.

[7]The number of cities included in this empirical analysis is mainly constrained by the availability of disposable income data.

Figure 5: The price-to-income ratios of 89 cities in China.




## 6.2 Implementation Details

We apply the PSY-factor procedure to the price-to-income ratios in each group. To implement the PSY test, we set the minimum window size to be 21 observations, based on the suggested rule in Phillips et al. (2015*a*), so the evolving test recursion begins in September 2004. The DF regression model in (3.3) is augmented with lags and lag order is selected by BIC with a maximum lag order setting of 4.

To account for potential heteroskedasticity in the monthly price-to-income ratios and the multiplicity issue of recursive testings, we use a composite bootstrap procedure for calculating critical values as developed in Phillips & Shi (2019).[8] The empirical size is 5%, controlled over a one-year period. Suppose $T_b$ is the number of observations in the control window. The probability of making at least one false positive rejection over the period with $T_b = 12$ observations is 5%. The procedure is detailed in full below.

**Step 1:** Estimate the regression model (3.3) under the restriction of $\beta = 0$ (null hypothesis) using $\tilde{y}_t$ (the first common factor estimated from the PIRs). The estimated coefficient and residuals are denoted, respectively, by $\hat{\alpha}$ and $e_t$.

**Step 2:** Generate a bootstrap sample with $T_0 + T_b - 1$ observations using the formula

$$\tilde{y}_t^b = \hat{\alpha} + \tilde{y}_{t-1}^b + e_t^b \tag{6.1}$$

with initial values $\tilde{y}_1^b = \tilde{y}_1$. The residual $e_t^b = w_t e_l$, where $w_t$ follows standard normal distribution and $e_l$ is bootstrapped from the residuals obtained in Step 1.

[8] In the simulation study, for simplicity we assume the errors are well behaved and follow a standard normal distribution. The bootstrap procedure is therefore replaced with a finite sample simulation to improve computational speed.

**Step 3:** Compute

$$\mathcal{M}_t^b = \max_{t \in [T_0, T_0+T_b-1]} \left( PSY_t^b \right),$$

from the bootstrap sample

**Step 4:** Repeat Steps 2-3 for $B = 5,000$ times.

**Step 5:** The 95% percentiles of the $\left\{ \mathcal{M}_t^b \right\}_{b=1}^B$ sequence serves as the critical value of the PSY-factor procedure.

## 6.3 Common Bubbles

Figure 6 presents the estimated first common components (black lines) and the identified bubble periods (green shaded areas). Overall, the two estimated first common components show similar dynamics but the fluctuations in group I are far more dramatic. There are three periods of rapid expansion in the first factor of both groups, which occur around 2004-2005, 2007-2008 and 2010.

We apply the PSY procedure to the estimated first factors. Consistent with our observations from the estimated factors, we find more evidence of common bubbles in the Group I cities than Group II. The PSY procedure suggests three explosive episodes in Group I and two explosive periods in Group II. The first episode occurs at the beginning of the sample period from 2004M10-2005M02 in Group I. This episode terminates one month earlier in Group II.

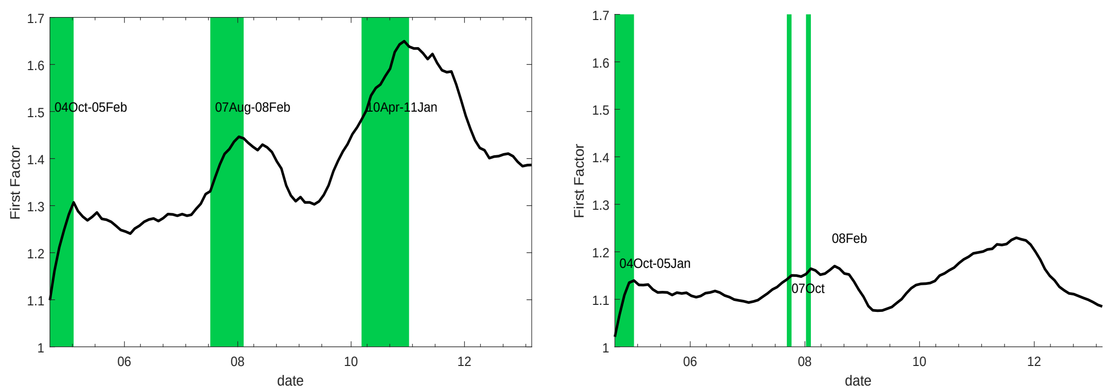
The second episode in Group I runs from August 2007 to February 2008. By comparison, the evidence of speculative behavior in the Group II markets is not as strong and only occurs in two months: October 2007 and February 2008. We observe an additional episode of speculation in Tier 1 and 2 cities from March 2010 to January 2011, whereas no evidence of speculation is detected in the Tier 3 housing markets over this period.

In addition, we consider a pseudo real-time implementation of the PSY-factor procedure on these real estate markets, i.e., using only information up to the observation of interest for estimation of the primary common factor in the first step. The identified bubble episodes are displayed in Figure 8 (Appendix D). For Group I, the identified episodes are almost identical to those in Figure 6(a), with one small exception: the starting date of the last episode is found to be three months later. For Group II, the first episode is exactly the same but no bubbles are detected around 2007-2008.

Interestingly, the identified collapse time of the first bubble episode is one month ahead of the "Circulation on Stabilising Housing Price" document (GOSC [2005] No. 8) issued by the General Office of the State Council (GOSC). This document (known as the old "Guo Ba Tiao") underlines the importance of housing price stability in the form of administrative

Figure 6: The identified bubble periods. The solid (black) lines are the estimated first factors from respective groups. The shaded (green) areas, with dates, show the periods when the PSY-factor test rejects the null hypothesis of a unit root against the explosive alternative for the first common factor.

(a) Tier 1 and 2 (b) Tier 3




accountability and is followed by a new "Guo Ba Tiao" document (GOSC [2005] No. 26) issued by seven ministries in May 2005. The new "Guo Ba Tiao" delivers a series of cooling-measure policies aimed at restraining housing demand, including raising the preferential mortgage interest rate from 5.31% to 6.12%, raising the down payment from 20% to 30%, and imposing a sales tax of 5.5% on the gross re-sale price for house owners who resold their houses within 2 years of occupancy. The government launched a second round of supply-side regulations and a foreign investment regulations in 2006 (GOSC [2006] No. 37). For example, on the supply side one of the regulations requires that at least 70% of newly registered or constructed units are to have floor areas no larger than 90 square meters and accelerating the construction of low cost rental housing for low income families; and on foreign investment, the new regulations restricted, inter alia, foreign purchases of apartments to institutions and individuals with established branches and residency.

The origination date of the second episode is one month before the "927 Housing Mortgage Policy" by the People's Bank of China (September 2007). This policy requires that the down payment for first home buyers be no lower than 20% for units less than 90 square meters and no lower than 30% for units above 90 square meters. For those who apply for a second loan, the down payment should not be lower than 40% and the interest rate for such a loan should not be lower than 1.1 times the benchmark interest rate.

The empirical findings on dating the emergence of explosive real estate market behavior match well the introduction of government housing policies designed to cool housing market prices. For instance, the identified origination date of the third episode is three months behind "The Circular on Promoting the Stable and Healthy Development of the Real Estate Market" (SC [2010] No. 4) issued by the State Council (January 2010), and actually coincides with

"The Notification Regarding the Steady and Healthy Development of the Real Estate Market" (SC [2010] No. 10). These two documents are followed by the "Notice of Issues Relating to Standardising Different Residential Mortgage Loan Policies" (MOHUR and MF [2010] No. 179) issued by the Ministry of housing and urban and rural development, Ministry of Finance, People's Bank of China and China Banking Regulatory Commission. Several measures were imposed in the sequence of documents to cool the impact on prices of rising housing demand. For example, the down payment for first home buyers was raised to 30%, at least 40% of total construction area was required to be allocated for affordable and moderate-sized units, and commercial banks were required to suspend loans to customers for the purchase of a third or subsequent residence.

In summary, some coincidence is observed between observed bubble behavior and enacted regulatory cooling policies. In particular, as the first bubble episode collapsed, the State Council issued a nation-wide cooling policy, which in this case was in retard of the market; then, soon after the second bubble origination, the central bank imposed further cooling measures; yet further measures were enacted three months before the third episode, showing continuing concerns by regulators of housing prices. In short, the Chinese government regulators became steadily more active and aggressive in implementing price-cooling measures throughout this period. The date-stamping mechanism for the presence of an exuberant factor in housing prices therefore helps to provide a context for the timing of government policies intended to reduce market exuberance.

# 7 Conclusion

Price bubbles in the financial system and asset markets such as those in real estate pose a significant threat to economic and financial stability. Such disturbances from normal market behavior have led to the introduction in many countries of macroprudential and microprudential policy regulations that are designed to moderate market behavior. In many cases, emergent speculative elements in financial and real estate asset markets are influenced by driving factors of the behavioral kind that are not directly observed. It is therefore particularly useful to have econometric methods that enable the detection of such behavior via the estimation and testing of the unobserved factors that may be driving speculative activity. Based on earlier methods in Phillips et al. (2015*a*,*b*) that were designed for observed data, this paper provides tools that enable such identification and empirical detection of an unobserved common explosive factor influencing market behavior coupled with a real-time mechanism for their dating and identification.

The factor methods developed here may be applied to large dimensional financial data sets and simulation results show good performance in the detection of unobserved common bubble factors in terms of test size, successful detection rates, and the estimation accuracy of bubble origination dates. The empirical application to real estate markets in major Chinese cities reveals strong evidence of a common driving factor affecting markets in the leading Tier 1 cities with three common bubble episodes identified in the periods 2004-2005, 2007-2008, and 2010-2011. Real time dating exercises show results that match well against government regulatory policies that were introduced as cooling measures to mitigate housing price bubble activity in the real estate market. Unobserved factor methods of the typed developed here therefore seem to offer some promise as a potential guide to regulatory authorities faced with emergent speculative behavior.

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# A Appendix A: Preliminary Lemmas

**Lemma A.1** *Under Assumption 4.1 and 4.3, as $T \to \infty$, we have the following:*

*(1)* $\sum_{t=1}^{T} e_{it-1} = O_p\left(T^{1/2}\right)$ *and* $\sum_{t=1}^{T}\left(e_{it} - e_{it-1}\right) = O_p\left(1\right)$;

*(2)* $\sum_{t=1}^{T} e_{it}e_{it-1} = O_p\left(T\right)$ *and* $\sum_{t=1}^{T} e_{it-1}^2 = O_p\left(T\right)$;

*(3)* $\sum_{t=1}^{T} u_{0,t} = O_p\left(T^{1/2}\right); \sum_{t=1}^{T} u_{0,t}^2 = O_p\left(T\right)$;

*(4)* $\sum_{t=1}^{T} u_{1,t} = O_p\left(T^{1/2}\right); \sum_{t=1}^{T} u_{1,t}^2 = O_p\left(T\right)$;

*(5)* $\sum_{t=[\tau_0 T]+1}^{T} u_{1,t}e_{it-1} = O_p\left(T\right).$

**Proof.** The results follow directly from Assumptions 4.1 and 4.3 by application of suitable laws of large numbers and central limit theory, as in Bai (2004). ■

**Lemma A.2** *Under Assumption 4.1 and 4.3, as $T \to \infty$ we have*

$$\frac{1}{T^2}\sum_{t=1}^{T} f_{0,t-1}^2 \Rightarrow \int_0^1 B\left(r\right)^2 dr;$$

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T} f_{0,t-1} \Rightarrow \int_0^1 B\left(r\right) dr;$$

$$\frac{1}{T}\sum_{t=1}^{T} f_{0,t-1}u_{0,t} \Rightarrow \int_0^1 B\left(r\right) dB\left(r\right);$$

*where $B(r)$ is a Brownian motion with variance $\sigma_{00}$.*

**Proof.** The proofs follow standard methods, e.g. Phillips (1987) and Phillips & Perron (1988). ■

**Lemma A.3** *Under Assumption 4.1, 4.4, and 4.3, as $N, T \to \infty$, we have*

(1) $f_{1,t} = O_p\left(T^{\eta/2}\rho_T^{t-T_r}\right)$ *for* $t \geq T_r$;

(2) $\sum_{t=T_0+1}^{T} f_{1,t-1}u_{0,t} = O_p\left(T^{\eta}\rho_T^{T-T_r}\right)$, $\sum_{t=T_0+1}^{T} f_{1,t-1}u_{1,t} = O_p\left(T^{\eta}\rho_T^{T-T_r}\right)$ *and* $\sum_{t=T_0+1}^{T} f_{1,t-1}e_{i,t} = O_p\left(T^{\eta}\rho_T^{T-T_r}\right)$

(3) $\sum_{t=T_0+1}^{T} f_{1,t-1} = O_p\left(T^{3\eta/2}\rho_T^{T-T_r}\right)$;

(4) $\sum_{t=T_0+1}^{T} f_{1,t-1}^2 = O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right)$;

(5) $\sum_{t=T_0+1}^{T} f_{1,t-1}f_{0,t-1} = O_p\left(T^{(1+3\eta)/2}\rho_T^{T-T_r}\right)$;

(6) $\sum_{t=T_0+1}^{T} f_{1,t-1}\left(e_{it} - e_{it-1}\right) = O_p\left(T^{\eta}\rho_T^{T-T_r}\right)$.

**Proof.** (1) By definition,

$$f_{1,t} = f_{1,T_r}\rho_T^{t-T_r} + \sum_{j=T_r+1}^{t} \rho_T^{t-j}u_j.$$

Since $f_{1,T_r} = O_p\left(T^{\eta/2}\right)$ and $\frac{1}{T^{\eta/2}}\sum_{j=T_r+1}^{t}\rho_T^{T_r-j}u_j \Rightarrow N_c = \mathcal{N}\left(0, \frac{\sigma_{11}}{2c}\right)$ from Lemma 4.2 of Phillips & Magdalinos (2007*b*),

$$\frac{f_{1,t}}{T^{\eta/2}\rho_T^{t-T_r}} = \frac{f_{1,T_r}}{T^{\eta/2}} + \frac{1}{T^{\eta/2}}\sum_{j=T_r+1}^{t}\rho_T^{T_r-j}u_j \Rightarrow c_0 + N_c,$$

where $c_0$ is a constant. Thus, $f_{1,t} = O_p\left(T^{\eta/2}\rho_T^{t-T_r}\right)$.

(2) This follows directly from Phillips & Magdalinos (2007*b*).

(3) Since $f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}$, we have

$$\sum_{t=T_0+1}^{T} f_{1,t} = \rho_T \sum_{t=T_0+1}^{T} f_{1,t-1} + \sum_{t=T_0+1}^{T} u_{1,t}.$$

It follows that

$$(1-\rho_T)\sum_{t=T_0+1}^{T} f_{1,t-1} = -f_{1,T} + f_{1,T_0} + \sum_{t=T_0+1}^{T} u_{1,t} = -f_{1,T}\left[1+o_p(1)\right]$$

since $f_{1,T_0} = O_p\left(T^{\eta/2}\rho_T^{T_0-T_r}\right)$ and $f_{1,T} = O_p\left(T^{\eta/2}\rho_T^{T-T_r}\right)$ from Lemma A.3 (1) and $\sum_{t=T_0+1}^{T} u_{1,t} = O_p\left(T^{1/2}\right)$ from Lemma A.1. Therefore,

$$\sum_{t=T_0+1}^{T} f_{1,t-1} = \frac{T^\eta}{c} f_{1,T}\left[1+o_p(1)\right] = O_p\left(T^{3\eta/2}\rho_T^{T-T_r}\right).$$

(4) By squaring and summing up the equation $f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}$ from 1 to $T$, we have

$$\begin{aligned}\sum_{t=T_0+1}^{T} f_{1,t-1}^2 &= \frac{1}{\rho_T^2-1}\left[f_{1,T}^2 - f_{1,T_r}^2 - \sum_{t=T_0+1}^{T} u_{1,t}^2 - 2\rho_T \sum_{t=T_0+1}^{T} f_{1,t-1}u_{1,t}\right] \\ &= \frac{1}{\rho_T^2-1} f_{1,T}^2\left[1+o_p(1)\right] = O_p(1)\end{aligned}$$

since $f_{1,T}^2 = O_p\left(T^\eta \rho_T^{2(T-T_r)}\right)$ and $\sum_{t=T_0+1}^{T} f_{1,t-1}u_{1,t} = O_p\left(T^\eta\rho_T^{T-T_r}\right)$ from Lemma A.3 (1)-(2), $f_{1,T_r} = O_p\left(T^{\eta/2}\right)$, $\sum_{t=T_0+1}^{T} u_{1,t}^2 = O_p(T)$ from Lemma A.1, and $\frac{1}{T^\eta}\frac{1}{\rho_T^2-1} = O_p(1)$. Therefore,

$$\sum_{t=T_0+1}^{T} f_{1,t-1}^2 = O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right).$$

(5) The sum of the cross product between $f_{1,t-1}$ and $f_{0,t-1}$ over $[T_0+1, T]$ is

$$\begin{aligned}\sum_{t=T_0+1}^{T} f_{1,t-1}f_{0,t-1} &= T^{(1+\eta)/2}\sum_{t=T_0+1}^{T} \frac{f_{1,t-1}}{T^{\eta/2}\rho_T^{t-T_r}}\frac{f_{0,t-1}}{T^{1/2}}\rho_T^{t-T_r} \\ &\le T^{(1+\eta)/2}\max_{t\in[T_0+1,T]}\left\{\frac{f_{1,t-1}}{T^{\eta/2}\rho_T^{t-T_r}}\right\}\max_{t\in[T_0+1,T]}\left\{\frac{f_{0,t-1}}{T^{1/2}}\right\}\sum_{t=T_0+1}^{T}\rho_T^{t-T_r} \\ &= O_p\left(T^{(1+3\eta)/2}\rho_T^{T-T_r}\right).\end{aligned}$$

(6) From Assumption 4.3, $\mathbb{E}\left(e_{it}-e_{it-1}\right) = 0$ and $Var\left(e_{it}-e_{it-1}\right) < \infty$. We have $\sum_{t=1}^{T} f_{1,t-1}\left(e_{it}-e_{it-1}\right) = O_p\left(T^\eta\rho_T^{T-T_r}\right)$, which follows directly from Phillips & Magdalinos (2007*b*). ■

## B Appendix B: Proofs Under the Null Hypothesis

**Lemma B.1** *Under the null specification of model (2.3) and Assumption 4.1, 4.2(1), 4.3, 4.4, and 4.5, we have:*

$$(1)\ \sum_{t=1}^{T}\tilde{y}_{t-1}=\sum_{t=1}^{T}H^0 f_{0,t-1}\left[1+o_p(1)\right]=O_p\left(T^{3/2}\right);$$

$$(2)\ \sum_{t=1}^{T}\tilde{y}_{t-1}^2=\left(H^0\right)^2\sum_{t=1}^{T}f_{0,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^2\right);$$

$$(3)\sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1}=\left(H^0\right)^2\sum_{t=1}^{T}f_{0,t-1}u_{0,t}\left[1+o_p(1)\right]=O_p(T);$$

$$(4)\ \sum_{t=1}^{T}\Delta\tilde{y}_t^2=\left(H^0\right)^2\sum_{t=1}^{T}u_{0,t}^2\left[1+o_p(1)\right]=O_p(T);$$

$$(5)\ \sum_{t=1}^{T}\Delta\tilde{y}_t=H^0\sum_{t=1}^{T}u_{0,t}\left[1+o_p(1)\right]=O_p\left(T^{1/2}\right).$$

**Proof.** (1) The quantity

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{y}_{t-1}=\frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(\tilde{y}_{t-1}-H^0 f_{0,t-1}\right)+H^0\frac{1}{T^{3/2}}\sum_{t=1}^{T}f_{0,t-1}.$$

By the Cauchy-Schwarz inequality and Lemma 4.1, we have

$$\left[\sum_{t=1}^{T}\left(\tilde{y}_{t-1}-H^0 f_{0,t}\right)\right]^2\leqslant\sum_{t=1}^{T}\left(\tilde{y}_{t-1}-H^0 f_{0,t}\right)^2=O_p\left(T\delta_{NT}^{-2}\right)$$

and hence

$$\frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(\tilde{y}_{t-1}-H^0 f_{0,t-1}\right)=O_p\left(T^{-1/2}\delta_{NT}^{-1}\right).$$

From Lemma A.2, $T^{-3/2}\sum_{t=1}^{T}f_{0,t-1}=O_p(1)$ and $H^0=O_p(1)$, from Lemma S.1 in the Online Supplement. Therefore,

$$\sum_{t=1}^{T}\tilde{y}_{t-1}=H^0\sum_{t=1}^{T}f_{0,t-1}\left[1+o_p(1)\right]=O_p\left(T^{3/2}\right).$$

(2) Similarly,

$$\frac{1}{T^2}\sum_{t=1}^{T}\tilde{y}_{t-1}^2=\frac{1}{T^2}\sum_{t=1}^{T}\left(\tilde{y}_{t-1}-H^0 f_{0,t-1}+H^0 f_{0,t-1}\right)^2$$

$$= \frac{1}{T^2} \sum_{t=1}^{T} \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right)^2 + \frac{2}{T^2} H^0 \sum_{t=1}^{T} \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right) f_{0,t-1} + \frac{1}{T^2} \left( H^0 \right)^2 \sum_{t=1}^{T} f_{0,t-1}^2 \tag{B.1}$$

By Lemma 4.1, the first term of (B.1) is $\frac{1}{T^2} \sum_{t=1}^{T} \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right)^2 = O_p \left( T^{-1} \delta_{NT}^{-2} \right)$. The second term is

$$\frac{1}{T^2} \left| H^0 \sum_{t=1}^{T} \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right) f_{0,t-1} \right| \leqslant \frac{1}{T^{1/2}} \left| H^0 \right| \left( \frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_{t-1} - H^0 f_{0,t-1} \right|^2 \right)^{1/2} \left( \frac{1}{T^2} \sum_{t=1}^{T} f_{0,t-1}^2 \right)^{1/2}$$
$$= O_p \left( T^{-1/2} \delta_{NT}^{-1} \right).$$

The last term is $O_p(1)$ from Lemma A.2 and Lemma S.1 in the Online Supplement. Combining the above we have

$$\sum_{t=1}^{T} \tilde{y}_{t-1}^2 = \left( H^0 \right)^2 \sum_{t=1}^{T} f_{0,t-1}^2 \left[ 1 + o_p(1) \right] = O_p \left( T^2 \right).$$

(3) The quantity

$$\begin{aligned}
\sum_{t=1}^{T} \Delta \tilde{y}_t \tilde{y}_{t-1} &= \sum_{t=1}^{T} \tilde{y}_t \tilde{y}_{t-1} - \sum_{t=1}^{T} \tilde{y}_{t-1}^2 \\
&= \sum_{t=1}^{T} \left( \tilde{y}_t - H^0 f_{0,t} \right) \tilde{y}_{t-1} + H^0 \sum_{t=1}^{T} f_{0,t} \tilde{y}_{t-1} - \left( H^0 \right)^2 \sum_{t=1}^{T} f_{0,t-1}^2 \left[ 1 + o_p(1) \right] \qquad \text{(B.2)} \\
&= \sum_{t=1}^{T} \left( \tilde{y}_t - H^0 f_{0,t} \right) \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right) + H^0 \sum_{t=1}^{T} \left( \tilde{y}_t - H^0 f_{0,t} \right) f_{0,t-1} \qquad \text{(B.3)} \\
&+ H^0 \sum_{t=1}^{T} f_{0,t} \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right) + \left( H^0 \right)^2 \sum_{t=1}^{T} f_{0,t} f_{0,t-1} - \left( H^0 \right)^2 \sum_{t=1}^{T} f_{0,t-1}^2.
\end{aligned}$$

The first component of (B.3) is

$$\left| \sum_{t=1}^{T} \left( \tilde{y}_t - H^0 f_{0,t} \right) \left( \tilde{y}_{t-1} - H^0 f_{0,t-1} \right) \right| \leq T \left( \frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_t - H^0 f_{0,t} \right|^2 \right)^{1/2} \left( \frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_{t-1} - H^0 f_{0,t-1} \right|^2 \right)^{1/2}$$
$$= O_p \left( T \delta_{NT}^{-2} \right),$$

using (4.1). Similarly, the second component of (B.3) is

$$\left| H^0 \sum_{t=1}^{T} \left( \tilde{y}_t - H^0 f_{0,t} \right) f_{0,t-1} \right| \leq T^{\frac{3}{2}} \left| H^0 \right| \left( \frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_t - H^0 f_{0,t} \right|^2 \right)^{1/2} \left( \frac{1}{T^2} \sum_{t=1}^{T} f_{0,t-1}^2 \right)^{1/2} = O_p \left( T^{\frac{3}{2}} \delta_{NT}^{-1} \right).$$

By the same argument, the third component of equation (B.3) is at most $O_p\left(T^{\frac{3}{2}}\delta_{NT}^{-1}\right)$. The fourth component is of order $O_p\left(T^2\right)$ since $\left|H^0\right| = O_p(1)$ and

$$\sum_{t=1}^{T} f_{0,t} f_{0,t-1} = \sum_{t=1}^{T} f_{0,t-1}^2 + \sum_{t=1}^{T} f_{0,t-1} u_{0t} = \sum_{t=1}^{T} f_{0,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^2\right).$$

The fifth component is $O_p\left(T^2\right)$. Therefore, we have

$$\begin{aligned}\sum_{t=1}^{T} \Delta\tilde{y}_t \tilde{y}_{t-1} &= \left[\left(H^0\right)^2 \sum_{t=1}^{T} f_{0,t} f_{0,t-1} - \left(H^0\right)^2 \sum_{t=1}^{T} f_{0,t-1}^2\right]\left[1 + o_p(1)\right] \\ &= \left(H^0\right)^2 \sum_{t=1}^{T} f_{0,t-1} u_{0,t}\left[1 + o_p(1)\right] = O_p(T).\end{aligned}$$

(4) The quantity

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T} \Delta\tilde{y}_t^2 = \frac{1}{T}\sum_{t=1}^{T} \tilde{y}_t^2 - \frac{2}{T}\sum_{t=1}^{T} \tilde{y}_t \tilde{y}_{t-1} + \frac{1}{T}\sum_{t=1}^{T} \tilde{y}_{t-1}^2 \\ &= \frac{1}{T}\left(H^0\right)^2 \sum_{t=1}^{T}\left(f_{0,t}^2 - 2 f_{0,t} f_{0,t-1} + f_{0,t-1}^2\right)\left[1 + o_p(1)\right] \\ &= \frac{1}{T}\left(H^0\right)^2 \sum_{t=1}^{T}\left(f_{0,t} - f_{0,t-1}\right)^2\left[1 + o_p(1)\right] = \frac{1}{T}\left(H^0\right)^2 \sum_{t=1}^{T} u_{0,t}^2\left[1 + o_p(1)\right] = O_p(1).\end{aligned}$$

using Lemma A.1.

(5) The quantity

$$\sum_{t=1}^{T} \Delta\tilde{y}_t = \sum_{t=1}^{T} H^0\left(f_{0,t} - f_{0,t-1}\right)\left[1 + o_p(1)\right] = H^0 \sum_{t=1}^{T} u_{0,t}\left[1 + o_p(1)\right] = O_p\left(T^{1/2}\right)$$

using Lemma A.1. ■

## Proof of Theorem 4.2

**Proof.** We first derive the limiting distribution of $T\widehat{\beta}$. The OLS estimator $\widehat{\beta}$ is

$$\widehat{\beta} = \frac{T\sum_{t=1}^{T} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=1}^{T} \Delta\tilde{y}_t \sum_{t=1}^{T} \tilde{y}_{t-1}}{T\sum_{t=1}^{T} \tilde{y}_{t-1}^2 - \left(\sum_{t=1}^{T} \tilde{y}_{t-1}\right)^2}. \tag{B.4}$$

The denominator of (B.4) is

$$T\sum_{t=1}^{T}\tilde{y}_{t-1}^{2}-\left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^{2}=\left(H^{0}\right)^{2}\left[T\sum_{t=1}^{T}f_{0,t-1}^{2}-\left(\sum_{t=1}^{T}f_{0,t-1}\right)^{2}\right]\left[1+o_{p}\left(1\right)\right]$$

using Lemma B.1. The numerator is

$$T\sum_{t=1}^{T}\Delta\tilde{y}_{t}\tilde{y}_{t-1}-\sum_{t=1}^{T}\Delta\tilde{y}_{t}\sum_{t=1}^{T}\tilde{y}_{t-1}=\left(H^{0}\right)^{2}\left[T\sum_{t=1}^{T}f_{0,t-1}u_{0,t}-\sum_{t=1}^{T}u_{0,t}\sum_{t=1}^{T}f_{0,t-1}\right]\left[1+o_{p}\left(1\right)\right].$$

Thus,

$$\begin{aligned}T\widehat{\beta} &= \frac{T\sum_{t=1}^{T}f_{0,t-1}u_{0,t}-\sum_{t=1}^{T}u_{0,t}\sum_{t=1}^{T}f_{0,t-1}}{T\sum_{t=1}^{T}f_{0,t-1}^{2}-\left(\sum_{t=1}^{T}f_{0,t-1}\right)^{2}}\left[1+o_{p}\left(1\right)\right]\\ &\Rightarrow \frac{\int_{0}^{1}B\left(r\right)dB\left(r\right)-B\left(1\right)\int_{0}^{1}B\left(r\right)dr}{\int_{0}^{1}B\left(r\right)^{2}dr-\left[\int_{0}^{1}B\left(r\right)dr\right]^{2}},\end{aligned} \tag{B.5}$$

where $B(\cdot)$ is Brownian motion with variance $\sigma_{00}$.

Next we find the limit distribution of $\hat{\alpha}$. The least square estimator of $\hat{\alpha}$ is

$$\hat{\alpha}=\frac{\left(\sum_{t=1}^{T}\tilde{y}_{t-1}^{2}\right)\left(\sum_{t=1}^{T}\Delta\tilde{y}_{t}\right)-\left(\sum_{t=1}^{T}\Delta\tilde{y}_{t}\tilde{y}_{t-1}\right)\left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)}{T\sum_{t=1}^{T}\tilde{y}_{t-1}^{2}-\left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^{2}}. \tag{B.6}$$

The denominator of (B.6) is identical to that of (B.4). The numerator is

$$\begin{aligned}&\left(\sum_{t=1}^{T}\tilde{y}_{t-1}^{2}\right)\left(\sum_{t=1}^{T}\Delta\tilde{y}_{t}\right)-\left(\sum_{t=1}^{T}\Delta\tilde{y}_{t}\tilde{y}_{t-1}\right)\left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)\\ &=\left(H^{0}\right)^{3}\left[\sum_{t=1}^{T}f_{0,t-1}^{2}\sum_{t=1}^{T}u_{0,t}-\sum_{t=1}^{T}f_{0,t-1}u_{0,t}\sum_{t=1}^{T}f_{0,t-1}\right]\left[1+o_{p}\left(1\right)\right]\end{aligned} \tag{B.7}$$

using Lemma B.1. Therefore, we have

$$\begin{aligned}T^{1/2}\hat{\alpha} &= T^{1/2}H^{0}\frac{\sum_{t=1}^{T}f_{0,t-1}^{2}\sum_{t=1}^{T}u_{0,t}-\sum_{t=1}^{T}f_{0,t-1}u_{0,t}\sum_{t=1}^{T}f_{0,t-1}}{T\sum_{t=1}^{T}f_{0,t-1}^{2}-\left(\sum_{t=1}^{T}f_{0,t-1}\right)^{2}}\left[1+o_{p}\left(1\right)\right]\\ &\Rightarrow H^{0}\frac{\int_{0}^{1}B\left(r\right)^{2}drB\left(1\right)-\int_{0}^{1}B\left(r\right)dB\left(r\right)\int_{0}^{1}B\left(r\right)dr}{\int_{0}^{1}B\left(r\right)^{2}dr-\left[\int_{0}^{1}B\left(r\right)dr\right]^{2}}\end{aligned}$$

using Lemma A.2. Since $H^{0}=O_{p}(1)$, we have

$$T^{1/2}\hat{\alpha}=O_{p}\left(1\right). \tag{B.8}$$

The estimated variance of $\widehat{\beta}$ is

$$var\left(\widehat{\beta}\right)=\widehat{\sigma}_v^2\left[\sum_{t=1}^{T}\tilde{y}_{t-1}^2-\frac{1}{T}\left(\sum\tilde{y}_{t-1}\right)^2\right]^{-1},$$

where $\widehat{\sigma}_v^2=\frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t-\hat{\alpha}-\hat{\beta}\tilde{y}_{t-1}\right)^2$ which can be written as

$$\begin{aligned}\widehat{\sigma}_v^2&=\frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t-\hat{\alpha}-\hat{\beta}\tilde{y}_{t-1}\right)^2\\&=\frac{1}{T}\sum_{t=1}^{T}\Delta\tilde{y}_t^2+\hat{\alpha}^2+\hat{\beta}^2\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}^2-2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\Delta\tilde{y}_t-2\hat{\beta}\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}\Delta\tilde{y}_t+2\hat{\alpha}\hat{\beta}\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}\\&=\frac{1}{T}\sum_{t=1}^{T}\Delta\tilde{y}_t^2\left[1+o_p(1)\right]=\frac{1}{T}\left(H^0\right)^2\sum_{t=1}^{T}u_{0,t}^2\left[1+o_p(1)\right].\end{aligned}$$

The first term dominates other terms since $\hat{\alpha}=O_p\left(T^{-1/2}\right)$, $\hat{\beta}=O_p\left(T^{-1}\right)$, and

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\Delta\tilde{y}_t^2&=O_p(1);\\\hat{\beta}^2\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}^2&=O_p\left(T^{-2}\right)O_p(T)=O_p\left(T^{-1}\right);\\2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\Delta\tilde{y}_t&=O_p\left(T^{-1/2}\right)O_p\left(T^{-1/2}\right)=O_p\left(T^{-1}\right);\\2\hat{\beta}\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}\Delta\tilde{y}_t&=O_p\left(T^{-1}\right)O_p\left(T^{-1}\right)O_p(T)=O_p\left(T^{-1}\right);\\2\hat{\alpha}\hat{\beta}\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{t-1}&=O_p\left(T^{-1/2}\right)O_p\left(T^{-1}\right)O_p\left(T^{1/2}\right)=O_p\left(T^{-1}\right).\end{aligned}$$

It follows that

$$\begin{aligned}T^2var\left(\widehat{\beta}\right)&=\frac{\widehat{\sigma}_v^2}{\frac{1}{T^2}\sum_{t=1}^{T}\tilde{y}_{t-1}^2-\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^2}\\&=\frac{\frac{1}{T}\sum_{t=1}^{T}u_{0,t}^2}{\frac{1}{T^2}\sum_{t=1}^{T}f_{0,t-1}^2-\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}f_{0,t-1}\right)^2}\left[1+o_p(1)\right]\\&\Rightarrow\frac{\sigma_{00}^2}{\int_0^1B(r)^2dr-\left(\int_0^1B(r)dr\right)^2}.\end{aligned}\tag{B.9}$$

Therefore, using (B.5) and (B.9), the DF test has the following limiting form statistic

$$DF=\frac{T\widehat{\beta}}{\left[T^{2}var\left(\widehat{\beta}\right)\right]^{1/2}}\Rightarrow\frac{\int_{0}^{1}W\left(r\right)dW\left(r\right)-W\left(1\right)\int_{0}^{1}W\left(r\right)dr}{\left\{\int_{0}^{1}W\left(r\right)^{2}dr-\left[\int_{0}^{1}W\left(r\right)dr\right]^{2}\right\}^{1/2}},$$

where $W\left(\cdot\right)$ is standard Brownian motion. ■

# C Appendix C: Proofs Under the Alternative

## Proof of Lemma 4.4

**Proof.** Since $\tilde{\xi}_t=\tilde{L}'X_t/N$ and $X_t=\Gamma g_t+e_t$, we have

$$\begin{aligned}\tilde{\xi}_t&=\tilde{L}'\left(\Gamma g_t+e_t\right)/N\\&=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_i\gamma_{i1}g_{1t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_i\gamma_{i2}g_{2t}++\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_i\gamma_{i3}g_{3t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_ie_{it},\\&=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_i\lambda_{2,i}g_{1t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_i\lambda_{0,i}g_{3t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_ie_{it}\end{aligned}$$

by construction (recall that $\Gamma_1=\Lambda_2$, $\Gamma_2=\Lambda_1=0_{N\times1}$, and $\Gamma_3=\Lambda_0$). The estimated first common factor

$$\begin{aligned}\tilde{y}_t&=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{2,i}g_{1t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{0,i}g_{3t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it}\\&=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{2,i}f_{1,t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{0,i}f_{0,t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it}\\&=a_{N,T}f_{1,t}+b_{N,T}f_{0,t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it},\end{aligned}\tag{C.1}$$

where $a_{N,T}:=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{2,i}$ and $b_{N,T}:=\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{0,i}$. For notation simplicity, we use $a$ and $b$ to denote $a_{N,T}$ and $b_{N,T}$, respectively, in the subsequent analysis. By Hölder's inequality, we have

$$\left|\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\lambda_{2,i}\right|\leqslant\frac{1}{N}\sum_{i=1}^{N}\left|\tilde{l}_{i1}\lambda_{2,i}\right|\leqslant\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}^{2}\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\lambda_{2,i}^{2}\right)^{1/2}=O_p\left(1\right),$$

since $\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}^{2}=O_p\left(1\right)$ from the normalization constraint, and $\frac{1}{N}\sum_{i=1}^{N}\lambda_{2,i}^{2}=O_p\left(1\right)$ from the Assumption 4.2. Thus, $a=O_p\left(1\right)$. By the same argument, we have $b=O_p\left(1\right)$, $\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}=O_p\left(1\right)$, and $\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it}=O_p\left(1\right)$. ■

**Lemma C.1** *Under the alternative (2.7) with Assumption 4.1, 4.2(2), 4.3, 4.4, and 4.5, we have:*

$$(1)\ \sum_{t=1}^{T}\tilde{y}_{t-1}=a\sum_{t=T_0+1}^{T}f_{1,t-1}\left[1+o_p(1)\right]=O_p\left(T^{3\eta/2}\rho_T^{T-T_r}\right);$$

$$(2)\ \sum_{t=1}^{T}\tilde{y}_{t-1}^2=a^2\sum_{t=T_0+1}^{T}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right);$$

$$(3)\ \sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1}=a^2\left(\rho_T-1\right)\sum_{t=T_0+1}^{T}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{\eta}\rho_T^{2(T-T_r)}\right);$$

$$(4)\ \sum_{t=1}^{T}\Delta\tilde{y}_t^2=a^2\frac{c^2}{T^{2\eta}}\sum_{t=T_0+1}^{T}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(\rho_T^{2(T-T_r)}\right);$$

$$(5)\sum_{t=1}^{T}\Delta\tilde{y}_t=a\left(\rho_T-1\right)\sum_{t=1}^{T}f_{1,t-1}\left[1+o_p(1)\right]=O_p\left(T^{\eta/2}\rho_T^{T-T_r}\right).$$

**Proof.** (1) From Lemma 4.4, we can rewrite $\sum_{t=1}^{T}\tilde{y}_{t-1}$ as

$$\sum_{t=1}^{T}\tilde{y}_{t-1}=a\sum_{t=T_0+1}^{T}f_{1,t-1}+b\sum_{t=1}^{T}f_{0,t-1}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\left(\sum_{t=1}^{T}e_{it-1}\right).$$

Using Lemma 4.4 and A.3, we know that

$$\sum_{t=1}^{T}\tilde{y}_{t-1}=a\sum_{t=T_0+1}^{T}f_{1,t-1}+O_p\left(T^{3/2}\right)=O_p\left(T^{3\eta/2}\rho_T^{T-T_r}\right). \tag{C.2}$$

(2) The quantity

$$\begin{aligned}\sum_{t=1}^{T}\tilde{y}_{t-1}^2&=\sum_{t=1}^{T}\left(af_{1,t-1}+bf_{0,t-1}+\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it-1}\right)^2\\&=a^2\left(\sum_{t=1}^{T}f_{1,t-1}^2\right)+b^2\left(\sum_{t=1}^{T}f_{0,t-1}^2\right)+\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it-1}\right)^2\\&+2ab\left(\sum_{t=1}^{T}f_{1,t-1}f_{0,t-1}\right)+2a\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\left(\sum_{t=1}^{T}f_{1,t-1}e_{it-1}\right)\\&+2b\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\left(\sum_{t=1}^{T}f_{0,t-1}e_{it-1}\right).\end{aligned}$$

Using Lemma 4.4 and A.3, we have

$$\sum_{t=1}^{T} \tilde{y}_{t-1}^2 = a^2 \sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{(1+3\eta)/2}\rho_T^{T-T_r}\right) = O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right). \qquad \text{(C.3)}$$

(3) Similarly,

$$\begin{aligned}
&\sum_{t=1}^{T} \Delta\tilde{y}_t\tilde{y}_{t-1} \\
&= \sum_{t=1}^{T}\left[a(\rho_T-1)f_{1,t-1} + bu_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}(e_{it}-e_{it-1})\right]\left(af_{1,t-1} + bf_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it-1}\right) \\
&= a^2(\rho_T-1)\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + ab(\rho_T-1)\sum_{t=T_0+1}^{T} f_{1,t-1}f_{0,t-1} + a(\rho_T-1)\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\left(\sum_{t=T_0+1}^{T} f_{1,t-1}e_{it-1}\right) \\
&+ ab\sum_{t=T_0+1}^{T} u_{0,t}f_{1,t-1} + b^2\sum_{t=1}^{T} u_{0,t}f_{0,t-1} + b\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\right)\left(\sum_{t=1}^{T} u_{0,t}e_{it-1}\right) \\
&+ a\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\right)\left[\sum_{t=T_0+1}^{T} f_{1,t-1}(e_{it}-e_{it-1})\right] + b\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\right)\left[\sum_{t=1}^{T} f_{0,t-1}(e_{it}-e_{it-1})\right] \\
&+ \sum_{t=1}^{T}\left[\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}(e_{it}-e_{it-1})\right]\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it-1}\right). \qquad \text{(C.4)}
\end{aligned}$$

Therefore, we have

$$\sum_{t=1}^{T} \Delta\tilde{y}_t\tilde{y}_{t-1} = a^2(\rho_T-1)\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{(1+\eta)/2}\rho_T^{T-T_r}\right) = O_p\left(T^{\eta}\rho_T^{2(T-T_r)}\right), \qquad \text{(C.5)}$$

using Lemma 4.4 and A.3.

(4) The quantity

$$\begin{aligned}
&\sum_{t=1}^{T} \Delta\tilde{y}_t^2 \\
&= \sum_{t=1}^{T}\left[a(f_{1,t}-f_{1,t-1}) + b(f_{0,t}-f_{0,t-1}) + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}(e_{it}-e_{it-1})\right]^2 \\
&= \sum_{t=1}^{T}\left[a\frac{c}{T^{\eta}}f_{1,t-1} + au_{1,t} + bu_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}(e_{it}-e_{it-1})\right]^2 \\
&= a^2\frac{c^2}{T^{2\eta}}\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + \sum_{t=1}^{T}\left[au_{1,t} + bu_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}(e_{it}-e_{it-1})\right]^2
\end{aligned}$$

$$+ 2a\frac{c}{T^{\eta}}\sum_{t=T_0+1}^{T} f_{1,t-1}\left[au_{1,t} + bu_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\left(e_{it} - e_{it-1}\right)\right]$$

$$= a^2\frac{c^2}{T^{2\eta}}\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{\eta/2}\rho_T^{(T-T_r)}\right) = O_p\left(\rho_T^{2(T-T_r)}\right)$$

using Lemma 4.4, A.3.

(6) The quantity

$$\sum_{t=1}^{T}\Delta\tilde{y}_t = \sum_{t=1}^{T}\left[a\left(f_{1,t} - f_{1,t-1}\right) + b\left(f_{0,t} - f_{0,t-1}\right) + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\sum_{t=1}^{T}\left(e_{it} - e_{it-1}\right)\right],$$

$$= a\left(\rho_T - 1\right)\sum_{t=1}^{T} f_{1,t-1} + a\sum_{t=1}^{T} u_{1,t} + b\sum_{t=1}^{T} u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}\sum_{t=1}^{T}\left(e_{it} - e_{it-1}\right)$$

$$= a\left(\rho_T - 1\right)\sum_{t=1}^{T} f_{1,t-1} + O_p\left(T^{1/2}\right) = O_p\left(T^{\eta/2}\rho_T^{T-T_r}\right)$$

from Lemma A.3. ∎

## Proof of Theorem 4.5

**Proof.** We first derive the limiting distribution of $T\widehat{\beta}$. The denominator of $\widehat{\beta}$ in (B.4) is

$$T\sum_{t=1}^{T}\tilde{y}_{t-1}^2 - \left(\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^2 = Ta^2\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{3\eta}\rho_T^{2(T-T_r)}\right) = O_p\left(T^{1+2\eta}\rho_T^{2(T-T_r)}\right).$$

since $\sum_{t=1}^{T}\tilde{y}_{t-1}^2 = O_p\left(T^{1+2\eta}\rho_T^{2(T-T_r)}\right)$ and $\sum_{t=1}^{T}\tilde{y}_{t-1} = O_p\left(T^{3\eta/2}\rho_T^{T-T_r}\right)$ from Lemma C.1. The numerator is

$$\begin{aligned} T\sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1} - \sum_{t=1}^{T}\Delta\tilde{y}_t\sum_{t=1}^{T}\tilde{y}_{t-1} &= T\sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1} + O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right) \\ &= \left(\rho_T - 1\right)a^2T\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right) \end{aligned}$$

since $T\sum_{t=1}^{T}\Delta\tilde{y}_t\tilde{y}_{t-1} = O_p\left(T^{1+\eta}\rho_T^{2(T-T_r)}\right)$ and $\sum_{t=1}^{T}\Delta\tilde{y}_t\sum_{t=1}^{T}\tilde{y}_{t-1} = O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right)$ from Lemma C.1. Therefore,

$$\hat{\beta} = \frac{\left(\rho_T - 1\right)Ta^2\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right)}{Ta^2\sum_{t=T_0+1}^{T} f_{1,t-1}^2\left[1 + o_p\left(1\right)\right]} = \left(\rho_T - 1\right) + O_p\left(T^{-1}\right).$$

Next we derive the order of magnitude of $\hat{\alpha}$. By definition, we have

$$\hat{\alpha} = \frac{\left(\sum_{t=1}^{T} \tilde{y}_{t-1}^2\right)\left(\sum_{t=1}^{T} \Delta\tilde{y}_t\right) - \left(\sum_{t=1}^{T} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=1}^{T} \tilde{y}_{t-1}\right)}{T\sum_{t=1}^{T} \tilde{y}_{t-1}^2 - \left(\sum_{t=1}^{T} \tilde{y}_{t-1}\right)^2}.$$

From (C.4), we have

$$\sum_{t=1}^{T} \Delta\tilde{y}_t\tilde{y}_{t-1} = a^2\left(\rho_T - 1\right)\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + ab\left(\rho_T - 1\right)\sum_{t=T_0+1}^{T} f_{1,t-1}f_{0,t-1} + O_p\left(T^{\eta}\rho_T^{(T-T_r)}\right).$$

The numerator is

$$\begin{aligned}
& \left(\sum_{t=1}^{T} \tilde{y}_{t-1}^2\right)\left(\sum_{t=1}^{T} \Delta\tilde{y}_t\right) - \left(\sum_{t=1}^{T} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=1}^{T} \tilde{y}_{t-1}\right) \\
= & \left[a^2\left(\sum_{t=1}^{T} f_{1,t-1}^2\right) + 2ab\left(\sum_{t=1}^{T} f_{1,t-1}f_{0,t-1}\right) + O_p\left(T^{\eta}\rho_T^{T-T_r}\right)\right]\left[\sum_{t=1}^{T} a\left(\rho_T - 1\right)f_{1,t-1} + O_p\left(T^{1/2}\right)\right] - \\
& \left[a^2\left(\rho_T - 1\right)\sum_{t=T_0+1}^{T} f_{1,t-1}^2 + ab\left(\rho_T - 1\right)\sum_{t=T_0+1}^{T} f_{1,t-1}f_{0,t-1} + O_p\left(T^{\eta}\rho_T^{(T-T_r)}\right)\right]\left[\sum_{t=1}^{T} af_{1,t-1} + O_p\left(T^{3/2}\right)\right] \\
= & a^2b\left(\rho_T - 1\right)\left(\sum_{t=1}^{T} f_{1,t-1}f_{0,t-1}\right)\sum_{t=1}^{T} f_{1,t-1} + O_p\left(T^{\eta+1/2}\rho_T^{T-T_r}\right) \\
= & O_p\left(T^{2\eta+\frac{1}{2}}\rho_T^{2(T-T_r)}\right)
\end{aligned}$$

using Lemma C.1. Therefore,

$$\hat{\alpha} = \frac{O_p\left(T^{2\eta+\frac{1}{2}}\rho_T^{2(T-T_r)}\right)}{O_p\left(T^{2\eta+1}\rho_T^{2(T-T_r)}\right)} = O_p\left(T^{-1/2}\right).$$

The error variance estimator $\widehat{\sigma}_v^2$ is

$$\begin{aligned}
\widehat{\sigma}_v^2 &= \frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\alpha} - \hat{\beta}\tilde{y}_{t-1}\right)^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\beta}\tilde{y}_{t-1}\right)^2 + \hat{\alpha}^2 - 2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\beta}\tilde{y}_{t-1}\right)
\end{aligned}$$

Let $\xi_t = au_{1,t} + bu_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it} - \left(1+\hat{\beta}\right)\frac{1}{N}\sum_{i=1}^{N}\tilde{l}_{i1}e_{it-1}$. Since $\hat{\beta} = c/T^{\eta} + O_p\left(T^{-1}\right)$, $\rho_T - 1 - \hat{\beta} = O_p\left(T^{-1}\right)$. The first term is

$$\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\beta}\tilde{y}_{t-1}\right)^2$$

$$\begin{aligned}
&= \sum_{t=1}^{T}\left[a\left(\rho_T - 1 - \hat{\beta}\right) f_{1,t-1} - \hat{\beta} b f_{0,t-1} + \xi_t\right]^2 \\
&= a^2\left(\rho_T - 1 - \hat{\beta}\right)^2 \sum_{t=1}^{T} f_{1,t-1}^2 + b^2\hat{\beta}^2 \sum_{t=1}^{T} f_{0,t-1}^2 + \sum_{t=1}^{T} \xi_t^2 \\
&\quad -2ab\left(\rho_T - 1 - \hat{\beta}\right)\hat{\beta}\sum_{t=1}^{T} f_{1,t-1} f_{0,t-1} + 2a\left(\rho_T - 1 - \hat{\beta}\right)\sum_{t=1}^{T} f_{1,t-1}\xi_t - 2b\hat{\beta}\sum_{t=1}^{T} f_{0,t-1}\xi_t \\
&= a^2\left(\rho_T - 1 - \hat{\beta}\right)^2 \sum_{t=1}^{T} f_{1,t-1}^2 + O_p(\rho_T^{T-T_r}) \\
&= O_p\left(T^{2\eta-2}\rho_T^{2(T-T_r)}\right).
\end{aligned}$$

The second term is $O_p\left(T^{-1}\right)$. The third term is

$$\begin{aligned}
-2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\beta}\tilde{y}_{t-1}\right) &= -2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\left[a\left(\rho_T - 1 - \hat{\beta}\right) f_{1,t-1} - b\hat{\beta} f_{0,t-1} + \xi_t\right] \\
&= -2a\left(\rho_T - 1 - \hat{\beta}\right)\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T} f_{1,t-1} + 2b\hat{\alpha}\hat{\beta}\frac{1}{T}\sum_{t=1}^{T} f_{0,t-1} - 2\hat{\alpha}\frac{1}{T}\sum_{t=1}^{T}\xi_t \\
&= O_p\left(T^{3\eta/2-5/2}\rho_T^{T-T_r}\right)
\end{aligned}$$

Therefore, the error variance is

$$\widehat{\sigma}_v^2 = \frac{1}{T}\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\alpha} - \hat{\beta}\tilde{y}_{t-1}\right)^2 = O_p\left(T^{2\eta-2}\rho_T^{2(T-T_r)}\right).$$

Accordingly, the asymptotic order of the variance of $\widehat{\beta}$ is

$$var\left(\widehat{\beta}\right) = \frac{\widehat{\sigma}_v^2}{\sum_{t=1}^{T}\tilde{y}_{t-1}^2 - \left(\frac{1}{T^{1/2}}\sum_{t=1}^{T}\tilde{y}_{t-1}\right)^2} = \frac{O_p\left(T^{2\eta-2}\rho_T^{2(T-T_r)}\right)}{O_p\left(T^{2\eta}\rho_T^{2(T-T_r)}\right)} = O_p\left(T^{-2}\right).$$

Therefore, the DF test statistic has the following asymptotic order

$$DF = \frac{\widehat{\beta}}{\left[var\left(\widehat{\beta}\right)\right]^{1/2}} = \frac{O_p\left(T^{-\eta}\right)}{O_p\left(T^{-1}\right)} = O_p\left(T^{1-\eta}\right).$$

■

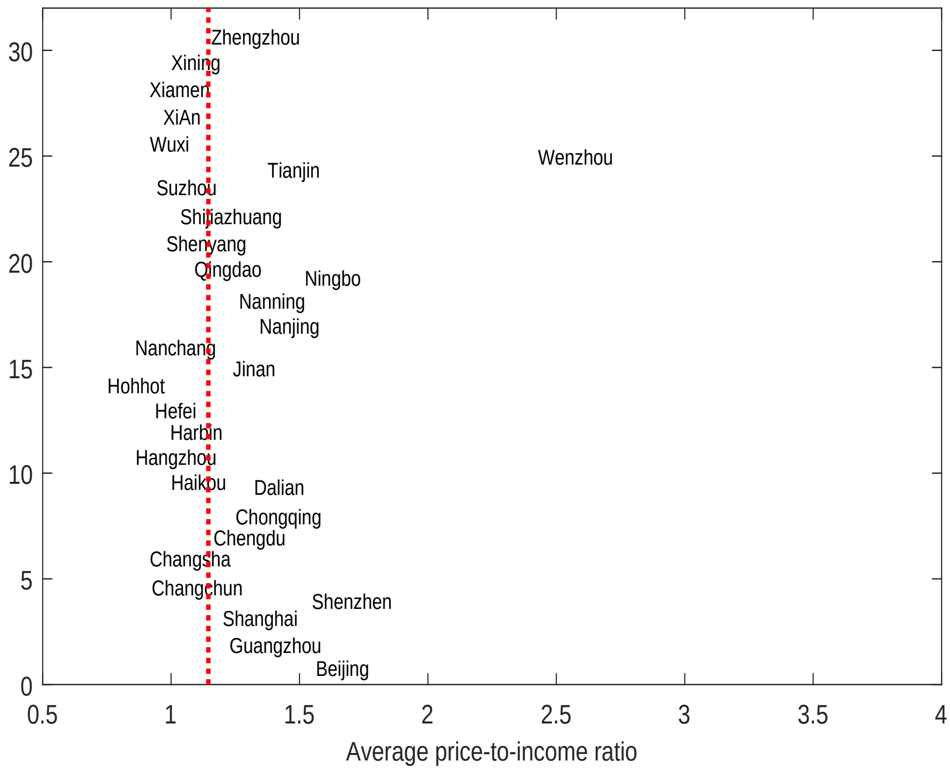
# Appendix D: Tables and Figures

Table 1: Tier 1, 2 and 3 cities

| | |
|---|---|
| **Tier 1** | Beijing, Shanghai, Guangzhou, Shenzhen |
| **Tier 2** | Changchun, Changsha, Chengdu, Chongqin, Dalian, Haikou, Hangzhou, Harbin, Hefei, Hohhot, Jinan, Nanchang, Nanjing, Ningbo, Qingdao, Shenyang, Shijiazhuang, Suzhou, Tianjin, Wenzhou, Wuxi, Xi'an, Xiamen, Xining, Zhengzhou |
| **Tier 3** | Anqing, Anshan, Baoding, Baotou, Bengbu, Changde, Changzhou, Chuzhou, Dandong, Deyang, Dongguan, Huai'an, Huzhou, Jianyan, Jiaxing, Jieyang, Jiujiang, Kaifeng, Langfang, Leshan, Lianyungang, Luohe, Luoyang, Luzhou, Mianyang, Nanchong, Nantong, Nanyang, Ningde, Qinhuang, Quanzhou, Rizhao, Shangrao, Shantou, Shaoxing, Songyuan, Suqian, Taizhou, Tangshan, Wuhu, Wuludao, Xingtai, Xuancheng, Xuzhou, Yancheng, Yangzhou, Yichun, Yingkou, Zaozhuang, Zhangjiakou, Zhangzhou, Zhaoqing, Zhenjiang, Zhongshan, Zhumadian |

Figure 7: The average price-to-income ratios of 89 cities in China. The vertical line indicates the national average price-to-income ratio over the sample period.

(a) Tier 1 and 2




(b) Tier 3

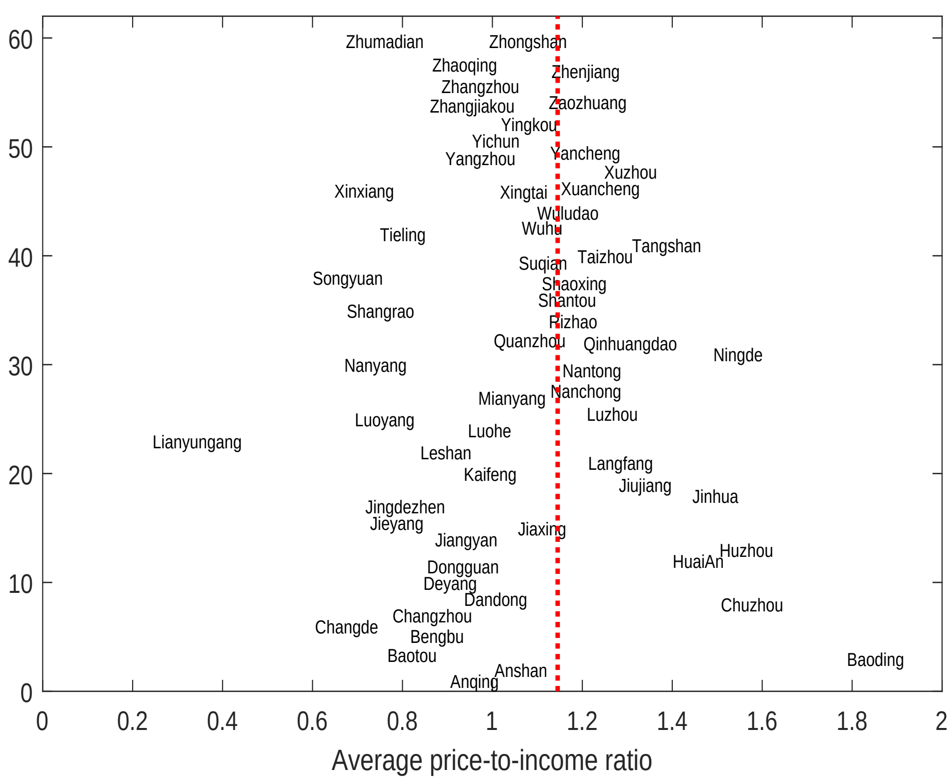
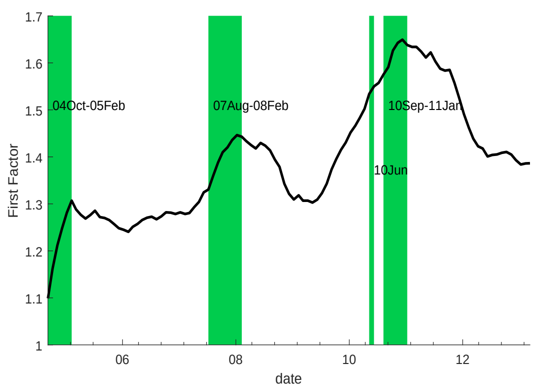


Figure 8: Pseudo real time identification of common bubbles. The black lines are the estimated first factors for the last observation of interest using the whole sample. The shaded (green) areas, shown with dates, are the periods where the PSY-factor test rejects the null hypothesis of a unit root against the explosive alternative for the first common factor.

(a) Tier 1 and 2




(b) Tier 3